New Foundations of the Theory of Elliptic Functions *

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Prooemium

Almost two years ago, when it pleased to examine the theory of elliptic functions in greater detail, I stumbled upon certain most important questions which seemed both to create a new branch of this theory and promote the analytic art significantly. Having given a satisfactory and because of the inherent difficulty hardly expected answer to those questions, I communicated the first major results, at first short and without a proof, then, because soon afterwards they seemed to be desired even more and, after new discoveries, those results seemed to be seen suspicious, with a proof with the geometers. At the same time, I was urged to publish the completed list of equations which I tackled. To satisfy this desire at least partly I decided to publish the foundations on which my questions are based. Now, we commend these new foundations of the theory of elliptic functions to the indulgence of the geometers.

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1 On the Transformation of Elliptic Functions

1.1 Exposition of the general Problem on the Transformation

1.

The most memorable integrals, which are exhibited by the formula $\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$, and which constitute the first kind of elliptic functions, as they are called, depend on two arguments, on the amplitude φ and the modulus k. Having compared the values of functions of this kind, which they obtain while the modulus stays fixed, the Analysists had detected many extraordinary things which concern their addition and multiplication. We, with great admiration, have recently seen, that this theory was pushed forward by Abel in his treatise (Crelle Journal für reine und angewandte Mathematik Vol. II).

Another question of not minor importance - understood in the broadest sense even involving the first - is the question on the comparison of the elliptic function for different moduli. After the beautiful discoveries of Legendre - the founder of the theory of elliptic functions - we, at first, reduced this question to certain principles and gave their general solution (Astronomische Nachrichten, 1827, n°123, 127). This theory on the transformation and all the things, which follow from this for the analysis of elliptic functions, we now want to explain in great detail.

2.

The general problem we want to tackle is the following:

" One looks for a rational function y of the element x of such a kind that we have

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}."$$

We see that this problem contains both the multiplication and the transformation.

Examples of rational functions y of such a kind satisfying the given problem are known for a long time. At first, it was known, no matter which odd number n was given, that one can exhibit a rational function y of such a kind that we have:

$$\frac{dy}{\sqrt{A + By + Cy^2 + Dy^3 + Ey^4}} = \frac{ndx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}};$$

this is the theorem on multiplication. For this aim, one has to assume this form:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(nn)}x^{nn}}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(nn)}x^{nn}},$$

after having determined the coefficients $a, a', a'', \cdots, ; b, b', b'', \cdots$ in the right manner. Sufficiently long it is even explored that this form

$$y = \frac{a + a'x + a''x^2}{b + b'x + b''x^2},$$

or this more general:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(2^m)}x^{2^m}}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(2^m)}x^{2^m}},$$

which arises from the preceding by iterated substitution, can be determined in such a way that it solves the problem. Recently, it was even proved by Legendre that for this aim this form, correctly determined, can be used:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(3^m)}x^{3^m}}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(3^m)}x^{3^m}}.$$

Connecting these two forms it is plain that the problem can be solved, after an appropriate choice for the coefficients, by putting:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(p)}x^p},$$

if p is a number of the form $2^\alpha 3^\beta (2m+1)^2$. Now, it will be proved in the following that the same holds, *no matter which number p is.*

1.2 The Principles of the Transformation

3.

Let us denote by U, V polynomial functions of the element x, further, let $y = \frac{U}{V}$, it is:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{VdU - UdV}{\sqrt{Y}},$$

having put for the sake of brevity:

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4.$$

The fraction $\frac{VdU-UdV}{\sqrt{Y}}$ can be transformed into a simpler form, as often as Y contains multiple factors; when except for four mutually different linear factors two of the remaining number are equal, the fraction, by itself, turns back into the differential of an elliptic function $\frac{dx}{M\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}}$, where M denotes a function of the element x. Let us examine the case in more detail and see, how often and which conditions it demands.

Let U, V be functions, the on of p-th order, the other of m-th order, such that $m \leq p$; Y will be of order $4p$. Now, that, having excluded the case of four linear factors, of the remaining factors of the function Y, whose number is $4p - 4$, two become equal to each other, $2p - 2$ conditions are to be satisfied. For, as many double linear factors the given function must have, as many conditional equations between its coefficients must hold.

But the functions U,V contain $m + p + 2$ or rather $m + p + 1$, because one of their total number can be set $= 1$, undetermined constant quantities. Therefore, their number either becomes equal to the number of $2p - 2$ conditions or the number of conditions is smaller than the number of undetermined quantities; let us suppose that m is any of the numbers $p - 3$, $p - 2$, $p - 1$,p in which cases the number of unknowns becomes $2p-2$, $2p-1$, $2p$, $2p+1$, respectively. It will be shown below that the first two cases are to be neglected and this is already plain by the following argument. For, having found the functions U,V, which provide the function Y with the prescribed form, if one puts $\alpha+\beta x$ instead of x, neither the structure of the functions U, V, Y nor the number of double factors of the function Y is changed: Hence, it is possible to introduce two arbitrary quantities from the beginning. Therefore, the number of undetermined quantities has to exceed the number of conditions by at least two units, whence the cases $m = p-3$ and $m = p-2$ are to be neglected. Further, we see, having put $\frac{\alpha+\beta x}{1+\gamma x}$ for x, that the third case can be reduced the fourth and the fourth is not changed by any means, in which case therefore three of the unknowns stay arbitrary and have to stay arbitrary.

Now, it is therefore shown, what can be concluded from the comparison of the number of undetermined quantities to the number of conditions: *Now matter what the number p is, the form:*

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(p)}x^p}$$

can be determined in such a way that we have:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{M\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}},$$

where M denotes a rational function of x: the solution can involve up to three arbitrary constants.

4.

In order to determine the function M, let

$$Y = (A + Bx + Cx^2 + Dx^3 + Ex^4)TT,$$

where T denotes a polynomial function of the element x: It will be

$$M = \frac{T}{V\frac{dU}{dx} - U\frac{dV}{dx}}.$$

T itself will be of order $2p-2$, and $V\frac{dU}{dx} - U\frac{dV}{dx}$ cannot be of higher order. Now, it is known in certain cases, of course whenever the number p has the form $2^\alpha 3^\beta (2n+1)^2$, that M becomes even constant. The same will be proved in the following, for every number p.

We can suppose that the functions U, V do not have a common factor; for, having assumed a common factor, the fraction $\frac{U}{V} = y$ is not changed. Let us resolve the expression

$$A' + B'y + C'y^2 + D'y^3 + E'y^4$$

into linear factors such that we have:

$$A' + B'y + C'y^2 + D'y^3 + E'y^4 = A'(1-\alpha'y)(1-\beta'y)(1-\gamma'y)(1-\delta'y),$$

whence it is:

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4 = A'(V-\alpha'U)(V-\beta'U)(V-\gamma'U)(V-\delta'U).$$

Now, there cannot exist a factor which is a common factor of all the quantities $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ or even only two of them; for, this factor would divide V and U at the same time, which we assumed to have no common factor. Therefore, where any linear factor divides the function Y twice, the same has to divide one of the quantities $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ and the same even twice.

Now, let us consider the following equations:

$$\begin{aligned}
(V-\alpha'U)\frac{dU}{dx} - \frac{V-\alpha'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx} \\
(V-\beta'U)\frac{dU}{dx} - \frac{V-\beta'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx} \\
(V-\gamma'U)\frac{dU}{dx} - \frac{V-\gamma'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx} \\
(V-\delta'U)\frac{dU}{dx} - \frac{V-\delta'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx},
\end{aligned}$$

from which it follows that a factor, which divides any of the quantities $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ and hence also its differential twice, also divides the expression $V\frac{dU}{dx} - U\frac{dV}{dx}$. But, we put the product conflated of all these factors, also dividing Y twice, $= T$, whence T will divide $V\frac{dU}{dx} - U\frac{dV}{dx}$. But, T is not of lower order than $V\frac{dU}{dx} - U\frac{dV}{dx}$, whence we see that

$$M = \frac{T}{V\frac{dU}{dx} - U\frac{dV}{dx}}$$

becomes a constant.

Additionally, we want to mention, if the one of the functions U, V would have been of lower order than $p-1$, that then also $V\frac{dU}{dx} - U\frac{dV}{dx}$ would have been of lower order than T, which nevertheless hast to divide the latter; since this is absurd, the cases $m = p-2$, $m = p-3$ must be neglected.

Therefore, it is now proved *that the form*

$$y = \frac{a + a'x + a''x^2 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + \cdots + b^{(p)}x^p},$$

for any number p, can be determined in such a way that the following identity results:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}},$$

This is the fundamental principle in the theory of transformations of elliptic functions.

1.3 It is propounded to reduce the Expression $\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$ to the simpler Form $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$.

5.

By of three arbitrary constants which we saw our solution to admit the expression $A + Bx + Cx^2 + Dx^3 + Ex^4$ can be transformed into this simpler one: $A(1-x^2)(1-k^2x^2)$. To illustrate this and the remaining things which were demonstrated by means of an example let the expression giben in the title be propounded:

$$\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$

which is, after having done the substitution

$$y = \frac{a + a'x + a''x^2}{b + b'x + b''x^2},$$

to be transformed into this simpler one

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

The question is about the determination of the substitution to be used and about the modulus k and the constant factor M from the given quantities α, β, γ, δ.

Let us put $a + a'x + a''x^2 = U$, $b + b'x + b''x^2 = V$, $y = \frac{U}{V}$; from the principles just explained it has to be:

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V) = K(1-x^2)(1-k^2x^2)(1+mx)^2(1+nx)^2,$$

where K denotes an arbitrary constant. Hence, we see that two from the total number of factors $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$, which will be of second order, even become squares. Therefore, let us put:

$$\begin{aligned} U-\gamma V &= C(1+mx)^2 \\ U-\delta V &= D(1+nx)^2. \end{aligned}$$

Concerning the remaining function $U-\alpha V, U-\beta V$, one can either put:

$$U-\alpha V = A(1-x^2), \qquad U-\beta V = B(1-k^2x^2)$$

or:

$$U-\alpha V = A(1-x)(1-kx), \quad U-\beta V = B(1+x)(1+kx),$$

where A, B, C, D denote constant quantities. The first possibility would have to be neglected. For, it will yield $\frac{U-\alpha V}{U-\beta V} = \frac{y-\alpha}{y-\beta} = \frac{A}{B} \cdot \frac{1-x^2}{1-k^2x^2}$, whence it would follow, having transformed x into $-x$, that y remains unchanged; that this is absurd is plain from the equations:

$$\begin{aligned} \frac{U-\alpha V}{U-\gamma V} &= \frac{y-\alpha}{y-\gamma} = \frac{A}{C} \cdot \frac{1-x^2}{(1+mx)^2} \\ \frac{U-\alpha V}{U-\delta V} &= \frac{y-\alpha}{y-\delta} = \frac{A}{D} \cdot \frac{1-x^2}{(1+nx)^2}. \end{aligned}$$

Therefore, one has to put:

$$\begin{array}{lll} (1) & U - \alpha V = & A(1-x)(1-kx) \\ (2) & U - \beta V = & A(1+x)(1+kx) \\ (3) & U - \gamma V & = C(1+mx)^2 \\ (4) & U - \delta V & = D(1+nx)^2. \end{array}$$

It should be mentioned that from the constants A, B, C, D one can be determined ad libitum.

6.

We see from equation (1), having put $x = 1$ and $x = \frac{1}{k}$, that $U = \alpha V$. Hence, from the equation:

$$\frac{U - \gamma V}{U - \beta V} = \frac{C}{B} \cdot \frac{(1+mx)^2}{(1+x)(1+kx)},$$

after having put $x = 1$, it arises:

$$\frac{\alpha - \gamma}{\alpha - \beta} = \frac{C}{B} \cdot \frac{(1+m)^2}{2(1+k)},$$

and for $x = \frac{1}{k}$:

$$\frac{\alpha - \gamma}{\alpha - \beta} = \frac{C}{B} \cdot \frac{(1+\frac{m}{k})^2}{2(1+\frac{1}{k})},$$

whence

$$(1+m)^2 = k\left(1 + \frac{m}{k}\right)^2.$$

Further, one will find in similar manner:

$$(1+n)^2 = k\left(1 + \frac{n}{k}\right)^2,$$

whence $m = \sqrt{k}, n = -\sqrt{k}$. Therefore, it is not possible to take m and n as equal; since then the expression $\frac{U-\gamma V}{U-\delta V} = \frac{y-\gamma}{y-\delta}$ and hence y would be a constant.

Now, in the equation

$$\frac{U-\gamma V}{U-\delta V}=\frac{y-\gamma}{y-\delta}=\frac{C}{D}\cdot\left\{\frac{1+\sqrt{k}\cdot x}{1-\sqrt{k}\cdot x}\right\}^2$$

let us at first put $x=1$ in which case $U=\alpha V$, and then $x=-1$ in which case $U=\beta V$: The following two equations arise:

$$\frac{\alpha-\gamma}{\alpha-\delta}=\frac{C}{D}\cdot\left\{\frac{1+\sqrt{k}}{1-\sqrt{k}}\right\}^2$$
$$\frac{\beta-\gamma}{\beta-\delta}=\frac{C}{D}\cdot\left\{\frac{1-\sqrt{k}}{1+\sqrt{k}}\right\}^2.$$

After having multiplied those equations by each other, we have:

$$\frac{C}{D}=\sqrt{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}},$$

whence it is possible to put:

$$C=\sqrt{(\alpha-\gamma)(\beta-\gamma)}$$
$$D=\sqrt{(\alpha-\delta)(\beta-\delta)};$$

for, one of the quantities A, B, C, D could be determined ad libitum.

From the same equations, having divided one by the other, we will obtain:

$$\frac{1+\sqrt{k}}{1-\sqrt{k}}=\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}}{\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}},$$

whence it follows:

$$\sqrt{k}=\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}.$$

Finally, let us note the formula:

$$\sqrt{k}+\frac{1}{\sqrt{k}}=2\cdot\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}},$$

whence we obtain:

$$
\begin{aligned}
(1-\sqrt{k})\left(1-\frac{1}{\sqrt{k}}\right) &= \frac{-4\sqrt{(\alpha-\delta)(\beta-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}} \\
(1+\sqrt{k})\left(1+\frac{1}{\sqrt{k}}\right) &= \frac{4\sqrt{(\alpha-\delta)(\beta-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}.
\end{aligned}
$$

In order to determine the constants A, B observe that from the equations $(1), (2), (3)$, after having put $x = \frac{1}{\sqrt{k}}$ leading to $U = \delta V$, it is found:

$$
\begin{aligned}
\frac{\delta-\alpha}{\delta-\gamma} &= \frac{A(1-\sqrt{k})\left(1-\sqrt{\frac{1}{k}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}} \\
\frac{\delta-\beta}{\delta-\gamma} &= \frac{B(1+\sqrt{k})\left(1+\sqrt{\frac{1}{k}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}},
\end{aligned}
$$

whence it follows:

$$
\begin{aligned}
A &= -\frac{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}{\gamma-\delta}\left\{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}\right\} \\
B &= \frac{\sqrt{(\beta-\gamma)(\beta-\delta)}}{\gamma-\delta}\left\{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}\right\}.
\end{aligned}
$$

7.

From the general principle established by us above it follows that in our example the expression $V\frac{dU}{dx} - U\frac{dV}{dx}$ will be equal to the product $(1+\sqrt{k}\cdot x)(1-\sqrt{k}\cdot x)$ multiplied by a constant which is proved by direct calculation.

It is, as it is plain from the expansion:

$$
(\gamma-\delta)\left(U\frac{dV}{dx}-V\frac{dU}{dx}\right) = (U-\gamma V)\frac{d(U-\delta V)}{dx} - (U-\delta V)\frac{d(U-\gamma V)}{dx}.
$$

But, we obtained:

$$\begin{aligned} U - \gamma V &= C(1+\sqrt{k}\cdot x)^2 \\ U - \delta V\ &= D(1-\sqrt{k}\cdot x)^2, \end{aligned}$$

whence it follows:

$$\begin{aligned} \frac{d(U-\gamma V)}{dx} &= \quad 2C(1+\sqrt{k}\cdot x)\sqrt{k} \\ \frac{d(U-\delta V)}{dx} &= -2D(1-\sqrt{k}\cdot x)\sqrt{k}. \end{aligned}$$

Hence, it arises:

$$(\gamma-\delta)\left(V\frac{dU}{dx}-U\frac{dV}{dx}\right) = 4\sqrt{k}\cdot CD(1+\sqrt{k}\cdot x)(1-\sqrt{k}\cdot x).$$

Having gathered all these in the right way we obtain:

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{4\sqrt{k}}{\gamma-\delta}\cdot\sqrt{\frac{CD}{-AB}}\cdot\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

whence it follows:

$$\begin{aligned} M = \frac{\gamma-\delta}{4\sqrt{k}}\sqrt{\frac{-AB}{CD}} &= \frac{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\delta)(\beta-\gamma)}}{4\sqrt{k}} \\ &= \left\{\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right\}^2, \end{aligned}$$

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$= \frac{dx}{\sqrt{[1-x^2]\left[\left(\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right)^4-\left(\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right)^4x^2\right]}}.$$

Having put $(\alpha-\gamma)(\beta-\delta)=G, (\alpha-\delta)(\beta-\gamma)=G'$ this becomes:

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{dx}{\sqrt{[1-x^2]\left[\left(\frac{\sqrt[4]{G}+\sqrt[4]{G'}}{2}\right)^4 - \left(\frac{\sqrt[4]{G}-\sqrt[4]{G'}}{2}\right)^4\right]}}.$$

Let $G = mm$, $G' = nn$, further let:

$$m' = \frac{1}{2}(m \ \ + n), \quad n' = \sqrt{mn}$$
$$m'' = \frac{1}{2}(m' + n'), \quad n'' = \sqrt{m'n'};$$

having put $x = \sin\varphi$ it will be:

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{d\varphi}{\sqrt{m''m''\cos^2\varphi + n''n''\sin^2\varphi}}.$$

Additionally, the value of x is easily calculated by means of the formula:

$$\frac{1-\sqrt{k}\cdot x}{1+\sqrt{k}\cdot x} = \sqrt[4]{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}}\cdot\sqrt{\frac{y-\delta}{y-\gamma}},$$

where:

$$\sqrt{k} = \frac{\sqrt[4]{G}-\sqrt[4]{G'}}{\sqrt[4]{G}+\sqrt[4]{G'}} = \sqrt[4]{\frac{m''m''-n''n''}{m''m''}}.$$

8.

The quantities $\alpha, \beta, \gamma, \delta$ in the propounded formulas can be interchanged ad libitum. This certainly is to our advantage and certainly, whenever a condition is added that, if it is possible, of course, the transformation succeeds by means of a real substitution. Let us examine this in more detail.

Let us put that the quantities $\alpha, \beta, \gamma, \delta$ are all real, further let $\alpha > \beta > \gamma > \delta$ such that $\alpha-\beta, \alpha-\gamma, \alpha-\delta$ are real positive quantities. Now, it has to be distinguished between the limits within which the value of y is contained:

1) δ and γ, 2) γ and β, 3) β and α, 4) α and δ.

In the last case, imagine that the transition from α to δ happens through infinity. We see that the expression

$$\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$

becomes real only in the second and fourth case, the expression

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$

becomes real only in the first and the third case. Table I indicates real substitutions which correspond to the four cases. The second table II contains the formulas, which serve for the transformation

$$\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)}}$$

into a simpler form by means of a substitution, for the limits within which the value of the argument y is contained:

1) $-\infty$ and γ, 2) γ and β, 3) β and α, 4) α and $+\infty$.

These formulas, by diving by $-\delta$ under the square root sign and putting $\delta = \infty$ then, can easily be derived from table I.

Table I.

(A.) $$\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} + \sqrt[4]{(\alpha-\beta)(\gamma-\delta)}}{2}, \quad N = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\beta)(\gamma-\delta)}}{2}$$

(I.) Limits: $\alpha \cdots \pm\infty \cdots \delta : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{(\alpha-\beta)(\beta-\delta)}{(\alpha-\gamma)(\gamma-\delta)}} \cdot \sqrt{\dfrac{y-\gamma}{y-\beta}}$

(II.) Limits: $\alpha \cdots\cdots\cdots\cdots \delta : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)}} \cdot \sqrt{\dfrac{\alpha-y}{y-\delta}}.$

$$(B.) \qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} + \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}, \quad N = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}$$

$$(\mathrm{I.}) \qquad \text{Limits:} \quad \beta \cdots\cdots \alpha : \frac{L-Nx}{L+Nx} = \sqrt[4]{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}} \cdot \sqrt{\frac{y-\delta}{y-\gamma}}$$

$$(\mathrm{II.}) \qquad \text{Limits:} \quad \delta \cdots\cdots \gamma : \frac{L-Nx}{L+Nx} = \sqrt[4]{\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}} \cdot \sqrt{\frac{\beta-y}{\alpha-y}}.$$

Table II.

$$(A.) \qquad \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{\alpha-\gamma} + \sqrt[4]{\alpha-\beta}}{2}, \quad N = \frac{\sqrt[4]{\alpha-\gamma} - \sqrt[4]{(\alpha-\beta}}{2}$$

$$(\mathrm{I.}) \qquad \text{Limits:} \quad \alpha \cdots \pm\infty : \frac{L-Nx}{L+Nx} = \sqrt[4]{\frac{\alpha-\beta}{\alpha-\gamma}} \cdot \sqrt{\frac{y-\gamma}{y-\beta}}$$

$$(\mathrm{II.}) \qquad \text{Limits:} \quad \gamma \cdots \pm\beta : \frac{L-Nx}{L+Nx} = \frac{\sqrt{\alpha-y}}{\sqrt[4]{(\alpha-\beta)(\alpha-\gamma)}}.$$

$$(B.) \qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{\alpha-\gamma} + \sqrt[4]{\beta-\gamma}}{2}, \quad N = \frac{\sqrt[4]{\alpha-\gamma} - \sqrt[4]{(\beta-\gamma}}{2}$$

$$(\mathrm{I.}) \qquad \text{Limits:} \quad \beta \cdots\cdots \alpha : \frac{L-Nx}{L+Nx} = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\gamma)}}{\sqrt{y-\gamma}}$$

$$(\mathrm{II.}) \qquad \text{Limits:} \quad -\infty \cdots \gamma : \frac{L-Nx}{L+Nx} = \sqrt[4]{\frac{\alpha-\gamma}{\beta-\gamma}} \cdot \sqrt{\frac{\beta-y}{\alpha-y}}.$$

In these formulas for the given limits, as x goes from -1 to 1 at the same time y goes from the one limit to the other. But, having commuted the limits corresponding to the formulas (I.) and (II.) we see that the expression $\frac{L-Nx}{L+Nx}$ creates an imaginary value of the form $\pm iR$, having put $i = \sqrt{-1}$ and where

R denotes a real quantity; additionally we see that x takes the form $\frac{Le^{i\varphi}}{N} = \frac{e^{i\varphi}}{\sqrt{k}}$, whence it follows

$$\frac{L - Nx}{L + Nx} = \frac{1 - e^{i\varphi}}{1 + e^{i\varphi}} = \frac{e^{-\frac{i\varphi}{2}} - e^{\frac{i\varphi}{2}}}{e^{-\frac{i\varphi}{2}} + e^{\frac{i\varphi}{2}}} = -i \tan \frac{\varphi}{2}.$$

Let us substitute the form to which we were lead in this occasion, $x = \frac{e^{i\varphi}}{\sqrt{k}}$, in the expression $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$. It arises:

$$\begin{aligned}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} &= \frac{ie^{i\varphi}d\varphi}{\sqrt{k} \cdot \sqrt{\left(1 - \frac{e^{2i\varphi}}{k}\right)\left(1 - e^{2i\varphi}\right)}} = \frac{d\varphi}{\sqrt{(1 - ke^{2i\varphi})(1 - ke^{-2i\varphi})}} \\ &= \frac{d\varphi}{\sqrt{1 - 2k\cos 2\varphi + kk}} = \frac{d\varphi}{\sqrt{(1-k)^2 \cos^2 \varphi + (1+k)^2 \sin^2 \varphi}}.\end{aligned}$$

This substitution certainly seems to be remarkable. For, by putting $x = \sin \psi$ from it is follows this even more general formula:

$$\frac{k^n \sin^{2n} \psi d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = \frac{(\cos 2n\varphi + i \sin 2n\varphi)d\varphi}{\sqrt{1 + 2k\cos 2\varphi + kk}},$$

whence one obtains for the limits 0 and π since the imaginary part vanishes:

$$\int\limits_0^\pi \frac{k^n \sin^{2n} \psi d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = \int\limits_0^\pi \frac{\cos 2n\varphi d\varphi}{\sqrt{1 - 2k\cos 2\varphi + kk}} = \int\limits_0^\pi \frac{\cos n\varphi d\varphi}{\sqrt{1 - 2k\cos \varphi + kk}},$$

which is a short proof of the remarkable formula given by Legendre. From the tables I. and II. it it possible to derive to others after having commuted the limits between which the valor of y is contained and having put $x = \frac{Le^{i\varphi}}{N}$. For the assigned limits the angle φ grows from 0 to π, whereas y goes from the one limit to the other.

Table III.

$$(A.)\qquad \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)}, \quad n = \frac{\sqrt{(\alpha-\gamma)(\beta-\gamma)} + \sqrt{(\alpha-\beta)(\gamma-\delta)}}{2}$$

$$\text{(I.)}\qquad \text{Limits:}\quad \gamma \cdots\cdots\cdots\cdots \beta: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\alpha-\beta)(\beta-\delta)}{(\alpha-\gamma)(\gamma-\delta)}} \cdot \sqrt{\frac{y-\gamma}{\beta-y}}$$

$$\text{(II.)}\qquad \text{Limits:}\quad \alpha \cdots \pm\infty \cdots \delta: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)}} \cdot \sqrt{\frac{y-\gamma}{y-\delta}}.$$

$$(B.)\qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\beta-\delta)(\alpha-\delta)(\beta-\gamma)}, \quad n = \frac{\sqrt{(\alpha-\gamma)(\beta-\delta)} + \sqrt{(\alpha-\delta)(\beta-\gamma)}}{2}$$

$$\text{(I.)}\qquad \text{Limits:}\quad \delta \cdots\cdots\cdots\cdots \gamma: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}} \cdot \sqrt{\frac{y-\delta}{\gamma-y}}$$

$$\text{(II.)}\qquad \text{Limits:}\quad \beta \cdots\cdots\cdots\cdots \alpha: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}} \cdot \sqrt{\frac{y-\beta}{\alpha-y}}.$$

Table IV.

$$(A.)\qquad \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\alpha-\beta)}, \quad n = \frac{\sqrt{\alpha-\gamma}+\sqrt{\alpha-\beta}}{2}$$

$$(\text{I.})\qquad \text{Limits:}\quad \gamma\cdots\cdots\beta: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{\alpha-\beta}{\alpha-\gamma}}\cdot\sqrt{\frac{y-\gamma}{\beta-y}}$$

$$(\text{II.})\qquad \text{Limits:}\quad \alpha\cdots+\infty: \quad \tan\frac{\varphi}{2} = \frac{\sqrt{y-\alpha}}{\sqrt[4]{(\alpha-\beta)(\alpha-\gamma)}}.$$

$$(B.)\qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\beta-\gamma)}, \quad n = \frac{\sqrt{\alpha-\gamma}+\sqrt{\beta-\gamma}}{2}$$

$$(\text{I.})\qquad \text{Limits:}\quad -\infty\cdots\gamma: \quad \tan\frac{\varphi}{2} = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\gamma)}}{\sqrt{\gamma-y}}$$

$$(\text{II.})\qquad \text{Limits:}\quad \beta\cdots\cdots\cdots\alpha: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{\alpha-\gamma}{\beta-\gamma}}\cdot\frac{y-\beta}{\alpha-y}.$$

We treated this question in more detail to have an elaborated example. The cases where either two or four of the quantities $\alpha, \beta, \gamma, \delta$ still remain. The first case also admits a real solution which is nevertheless free of the species of the imaginary. The second case does not allows such a solution at all. Therefore, to reduce everything to real numbers a new transformation will be necessary. whence the desired beauty of the formulas gets lost. Therefore, we will not address this question.

To the propounded substitution corresponds another formula, inverse to it, of the form

$$x = \frac{a + a'y + a''y^2}{b + b'y + b''y^2},$$

which yields most elegant formulas itself. But, that it does not seem that we might stay at this question too long, we want to postpone the investigation to another occasion. We return to general questions.

1.4 On the Transformation of the Expression $\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}$ into another similar one $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$

10.

We saw that the given expression

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}}$$

by means of a transformation of this kind:

$$y = \frac{a + a'x + a''x^2 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + \cdots + b^{(p)}x^p} = \frac{U}{V},$$

not matter what number p is, can be transformed into another similar to it:

$$\frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}.$$

A substitution of such a kind depends on the coefficients A', B', C', D', E' and crucially on the number p which denotes the exponent of the highest order found in the rational functions U, V. Therefore, we will in the following we will say that a substitution or transformation *is of $p-th$ order belongs to the $p-th$ order or, simpler, corresponds to the number p.*

Now, planning to examine the nature of this substitution in more detail, let us leave that more complex form:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + D'y^4}},$$

and let us discuss, how to transform this simpler form $\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}$ to which - as we saw and as it is known - the latter can be reduced, into another similar one $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$.

Having examined the nature of the propounded equation carefully it is found that the problem is solved if one of the functions U, V is odd and the other even; this is already indicated by the examples explored by the analysts up to now. In this task, it has to be distinguished between the case in which the order of the odd function is smaller and the case in which the order of the even is smaller, and the case in which the transformation belongs to an even number and the case in which the transformation is of odd order.

Now, let us therefore prove *at first* prove that the transformation succeeds having used a transformation of even order or of the form:

$$y = \frac{x(a + a'x^2 + a''x^4 + a^{(m-1)}x^{2m-2})}{a + b'x^2 + b''x^4 + b^{(m)}x^{2m}} = \frac{U}{V}.$$

Here, the functions $V + U, V - U, V + \lambda U, V - \lambda U$ will themselves all be of even order, whence we want to put:

$$\begin{aligned}
&(1.) \qquad V + U &&= (1 + x)(1 + kx)AA \\
&(2.) \qquad V - U &&= (1 - x)(1 - kx)BB \\
&(3.) \qquad V + \lambda U &&= CC \\
&(4.) \qquad V + \lambda U &&= DD,
\end{aligned}$$

where A, B, C, D denote polynomial functions of the element x. Those equations will be satisfied at the same time, it will be found as we proved:

$$\frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

Since, having changed x into $-x$, U becomes $-U$, but V is not changed, from the equations $(1.), (3.)$ the equations $(2.), (4.)$ follow directly. To satisfy equations $(1.), (3.)$, $V + \lambda U$ must have m times but $V + U$ must have $m - 1$ times two equal linear factors; in addition, $V + U$ must contain the factor $1 + x$. All these demand in total $m + m - 1 + 1 = 2m$ conditional equations which is the number of unknowns $a, a', \cdots, a^{(m-1)}; b', b'', \cdots b^{(m)}$. Hence, the propounded problem is determined.

Secondly, we want to prove that the transformation also succeeds having used a substitution of this kind:

$$y = \frac{x(a + a'x^2 + a''x^4 + a^{(m)}x^{2m})}{a + b'x^2 + b''x^4 + b^{(m)}x^{2m}} = \frac{U}{V},$$

which belong to an odd number. Here, $V+U, V-U, V+\lambda U, V-\lambda U$ are all themselves of odd order, whence we want to put:

$$\begin{aligned}
&(1.) \quad V+U &&= (1+x)AA \\
&(2.) \quad V-U &&= (1-x)BB \\
&(3.) \quad V+\lambda U &&= (1+kx)CC \\
&(4.) \quad V-\lambda U &&= (1-kx)DD.
\end{aligned}$$

Also here, only equations $(1.), (3.)$ will have to be satisfied, from which by changing x into $-x$ the remaining two follow directly. To satisfy those equations, it is necessary that both $V+U$ and $V+\lambda U$ m-times have two equal linear factors, for which aim $2m$ conditional equations will have to be satisfied; additionally $U+V$ has to obtain the factor $1+x$. Hence, we see that the number of conditional equations is $2m+1$ which is the number of unknowns $a, a', a'', \cdots, a^{(m)}; b', b'', \cdots, b^{(m)}$. Therefore, the problem is also well-defined in this case.

11.

Let us denote by U', V' polynomial functions of the element y of such a kind that, having put $z = \frac{U'}{V'}$, it is found:

$$\frac{dz}{\sqrt{(1-z^2)(1-\mu^2 z^2)}} = \frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}}.$$

Let the substitution that was done, $z = \frac{U'}{V'}$, be of $p'-$th order and by means of another substitution $y = \frac{U}{V}$ (where U, V as above denote polynomial functions of the element x) which we assume to be of order p, let us find as above:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Now, having substituted the value $y = \frac{U}{V}$ in the expression $z = \frac{U'}{V'}$, let $z = \frac{U''}{V''}$ be the result: Hence, the one substitution $z = \frac{U''}{V''}$ by means of which it is found:

$$\frac{dz}{\sqrt{(1-z^2)(1-\mu^2 z^2)}} = \frac{dx}{MM'\sqrt{(1-x^2)(1-k^2x^2)}},$$

will be of order pp'. So, we see that from several successively used transformations belonging to the numbers $p, p', p'', \cdots$ one can compose another one belonging to the number $pp'p'' \cdots$. And, vice versa it is always possible - what we will not prove here - to compose a transformation which has the composite order $pp'p'' \cdots$ from other successively applied ones which have order $p, p', p'' \cdots$, respectively. Therefore, it is only necessary to investigate those transformations that belong to the *first* order from which all the others can be constructed. Now, in the following let us therefore leave the first case which concerns the even order of transformation which can always be constructed from a transformation of odd order and a sufficiently often repeated transformation belonging to order 2. But let us examine the *second* case or the transformation of odd order in more detail now.

12.

We see that in this case we have to determine two functions, the one, V, of even order $2m$, the other, U, of odd order $2m+1$, in such a way that it holds:

$$V + U = (1+x)AA, \quad V + \lambda U = (1+kx)CC.$$

Now, I claim, if the functions U, V were determined in such a way that, having put $\frac{1}{kx}$ for x, $y = \frac{U}{V}$ becomes $\frac{1}{\lambda y} = \frac{V}{\lambda U}$, that then those equations follow from each other directly.

Let us put $V = \varphi(x^2), U = xF(x^2)$; we see that the expression $y = \frac{xF(x^2)}{\varphi(x^2)}$, having put $\frac{1}{kx}$ for x, goes into

$$\frac{F\left(\frac{1}{k^2x^2}\right)}{kx\varphi\left(\frac{1}{k^2x^2}\right)} = \frac{x^{2m}F\left(\frac{1}{k^2x^2}\right)}{kx \cdot x^{2m}\varphi\left(\frac{1}{k^2x^2}\right)},$$

where $x^{2m}F\left(\frac{1}{k^2x^2}\right), x^{2m}\varphi\left(\frac{1}{k^2x^2}\right)$ are polynomial functions. To make this equal to the expression $\frac{1}{\lambda y} = \frac{V}{\lambda U} = \frac{\varphi(x^2)}{\lambda x F(x^2)}$, the following equations must hold

$$\varphi(x^2) = px^{2m}F\left(\frac{1}{k^2x^2}\right), \quad \lambda F(x^2) = pkx^{2m}\varphi\left(\frac{1}{k^2x^2}\right),$$

where p denotes a constant quantity. If we in this equations put $\frac{1}{kx}$ for x, we obtain; $\varphi\left(\frac{1}{k^2x^2}\right) = \frac{p}{k^{2m}x^{2m}}F(x^2)$ and $\lambda F\left(\frac{pk}{k^2x^2}\right) = \frac{pk}{k^{2m}x^{2m}}\varphi(x^2)$. Comparing these to the first equations we get $\frac{p}{k^{2m}} = \frac{\lambda}{pk}$, whence $p = \sqrt{\lambda k^{2m-1}}$. Therefore, it follows:

$$\varphi(x^2) = \sqrt{\lambda k^{2m-1}} x^{2m} F\left(\frac{1}{k^2x^2}\right), \quad F(x^2) = \sqrt{\frac{k^{2m+1}}{\lambda}} x^{2m} \varphi\left(\frac{1}{k^2x^2}\right);$$

Of these equations the one follows from the other.

Now, as often as the expression

$$\frac{V+U}{1+x} = \frac{\varphi(x^2) + xF(x^2)}{1+x}$$

is a square of a polynomial function of the element x, the same will also hold for the other equation which is derived from the first by putting $\frac{1}{kx}$ for x and multiplying by $\sqrt{\lambda k^{2m-1}} x^{2m}$. Having done this we obtain, *if* $\frac{V+U}{1+x}$ *is a square, that the function:*

$$\begin{aligned}\sqrt{\lambda k^{2m-1}} x^{2m} \frac{\varphi\left(\frac{1}{k^2x^2}\right) + \frac{1}{kx} F\left(\frac{1}{k^2x^2}\right)}{1+\frac{1}{kx}} &= \frac{\sqrt{\lambda k^{2m-1}} x^{2m} F\left(\frac{1}{k^2x^2}\right) + \sqrt{\lambda k^{2m+1} x^{2m+1}} \varphi\left(\frac{1}{k^2x^2}\right)}{1+kx} \\ &= \frac{\varphi(x^2) + \lambda x F(x^2)}{1+kx} = \frac{V+\lambda U}{1+kx}\end{aligned}$$

will itself be a square. Q.D.E.

Therefore, the problem was reduced to the other problem that the expression

$$\frac{\varphi(x^2) + \sqrt{\frac{k^{2m+1}}{\lambda}} x^{2m+1} \varphi\left(\frac{1}{k^2x^2}\right)}{1+x} = \frac{V+U}{1+x}$$

is made a square where $\varphi(x^2)$ denotes an expression of this kind:

$$\varphi(x^2) = V = 1 + b' + b''x^4 + \cdots + b^{(m)}x^{2m}.$$

But, having put $U = xF(x^2) = x(a + a'x^2 + a''x^4 + \cdots + a^{(m)}x^{2m})$, because it is $U = xF(x^2) = \sqrt{\frac{k^{2m+1}}{\lambda}} x^{2m+1} \varphi\left(\frac{1}{k^2x^2}\right)$, we have:

$$\begin{aligned}(*) \qquad a &= \sqrt{\frac{k}{\lambda}} \cdot \frac{b^{(m)}}{k^m}, & a' &= \sqrt{\frac{k}{\lambda}} \cdot \frac{b^{(m-1)}}{k^{m-2}}, & a'' &= \sqrt{\frac{k}{\lambda}} \cdot \frac{b^{(m-2)}}{k^{m-4}}, \cdots \\ a^{(m)} &= \sqrt{\frac{k}{\lambda}} \cdot k^m & a^{(m-1)} &= \sqrt{\frac{k}{\lambda}} \cdot b'k^{m-2}, & a^{(m-2)} &= \sqrt{\frac{k}{\lambda}} \cdot b''k^{m-4} \cdots\end{aligned}$$

Now, we will come to some examples.

13.

Let $m = 1$ which is the simplest case, further let $V = 1 + b'x^2, U = x(a + a'x^2)$. Having put $A = 1 + \alpha x$ we find:

$$AA = (1 + \alpha x)^2 = 1 + 2\alpha x + \alpha\alpha x^2,$$

whence:

$$V + U = (1 + x)AA = 1 + (1 + 2\alpha)x + \alpha(2 + \alpha)x^2 + \alpha\alpha x^3.$$

Hence, it becomes:

$$b' = \alpha(2 + \alpha), \quad a = (1 + 2\alpha), \quad a' = \alpha\alpha.$$

The equations $(*)$ in § 12 become the following:

$$a = \sqrt{\frac{k}{\lambda}} \cdot \frac{b'}{k}, \quad a' = \sqrt{\frac{k^3}{\lambda}},$$

whence we obtain:

$$1 + 2\alpha = \frac{\alpha(2 + \alpha)}{\sqrt{k\lambda}}, \quad \alpha\alpha = \sqrt{\frac{k^3}{\lambda}}, \quad \alpha = \sqrt[4]{\frac{k^3}{\lambda}}.$$

Put $\sqrt[4]{k} = u, \sqrt[4]{\lambda} = v$, it will be $\alpha = \frac{u^3}{v}$, $1 + 2\alpha = \frac{v+2u^3}{v}, \alpha(2 + \alpha) = \frac{u^3(2v+u^3)}{v^2}$. Hence, the equation:

$$1 + 2\alpha = \frac{\alpha(2 + \alpha)}{\sqrt{k\lambda}}$$

turn into the following:

$$\frac{v + 2u^3}{v} = \frac{u(2v + u^3)}{v^4},$$

or

$$(1.) \qquad u^4 - v^4 + 2uv(1 - u^2v^2) = 0.$$

Additionally, it is

$$
\begin{aligned}
a\ &= 1+2\alpha = \frac{v+2u^3}{v}\\
a' &= \alpha\alpha = \frac{u^6}{v^2}\\
b' &= \alpha(2+\alpha) = \frac{u^3(2v+u^3)}{v^2} = vu^2(v+2u^3).
\end{aligned}
$$

From this we obtain:

$$
(2.) \qquad y = \frac{(v+2u^3)vx + u^6x^3}{v^2 + v^3u^2(v+2u^3)x^2}.
$$

Furthermore, we obtain because $1+y = \frac{(1+x)AA}{V}$:

$$
\begin{aligned}
&(3.) \qquad 1+y &&= \frac{(1+x)((v+u^3x)^2}{v^2+v^3u^2(v+2u^3)x^2}\\
&(4.) \qquad 1-y &&= \frac{(1-x)(v-u^3x)^2}{v^2+v^3u^2(v+2u^3)x^2}\\
&(5.) \qquad \sqrt{\frac{1-y}{1+y}} &&= \sqrt{\frac{1-x}{1+x}}\cdot\frac{v-u^3x}{v+u^3x}\\
&(6.) \qquad \sqrt{1-y^2} &&= \frac{\sqrt{1-x^2}(v^2-u^6x^2)}{v^2+v^3u^2(v+2u^3)x^2}.
\end{aligned}
$$

Further, by putting $\frac{1}{kx} = \frac{1}{u^4x}$ for x, since y becomes $\frac{1}{\lambda y} = \frac{1}{v^4y}$, we find a system of the following equations:

$$
\begin{aligned}
&(7.) \qquad 1+v^4y &&= \frac{(1+u^4x)(1+uvx)^2}{1+vu^2(v+2u^3)x^2}\\
&(8.) \qquad 1-v^4y &&= \frac{(1-u^4x)(1-uvx)^2}{1+vu^2(v+2u^3)x^2}\\
&(9.) \qquad \sqrt{\frac{1-v^4y}{1+v^y}} &&= \sqrt{\frac{1-u^4x}{1+u^4x}}\cdot\frac{1-uvx}{1+uvx}\\
&(10.) \qquad \sqrt{1-v^8y^2} &&= \frac{\sqrt{1-u^8x^2}(1-u^2v^2x^2)}{1+vu^2(v+2u^3)x^2}.
\end{aligned}
$$

14.

Having put

$$V+U=(1+x)AA, \quad V+\lambda U=(1+kx)CC,$$
$$V-U=(1-x)BB, \quad V-\lambda U=(1-kx)DD,$$

we see that it becomes:

$$ABCD=M\left(V\frac{dU}{dx}-U\frac{dV}{dx}\right),$$

where M denotes a constant quantity which can be found from the comparison of the coefficients in the expressions $ABCD, V\frac{dU}{dx}-U\frac{dV}{dx}$. Now, having put $V=b+b'x^2$+etc., $U=ax+a'x^3$+etc. in the expressions A,B,C,D the constant term becomes $\sqrt{b}$, whence we see that in the product of all of them the constant becomes bb, but in the expression $V\frac{dU}{dx}-U\frac{dV}{dx}$ the constant becomes ab, whence:

$$M=\frac{b}{a}.$$

Therefore, in our example, because $b=1, a=\frac{v+2u^3}{v}=\frac{u(2v+u^3)}{v^4}$, it is:

$$M=\frac{v}{v+2u^3}=\frac{v^4}{u(2v+u^3},$$

whence:

$$\frac{dy}{\sqrt{(1-y^2)(1-v^8y^2)}}=\frac{v+2u^3}{v}\cdot\frac{dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

The moduli k,λ which we saw to depend on each other by means of an equation of fourth order in § 13 (1.) are easily expressed rationally by the same quantity α. For, from the formulas given above:

$$\alpha=\frac{u^3}{v}, \quad 1+2\alpha=\frac{\alpha(2+\alpha)}{\sqrt{k\lambda}}=\frac{\alpha(2+\alpha)}{u^2v^2}$$

it follows:

$$\alpha=\frac{u^3}{v}, \quad u^2v^2=\frac{\alpha(2+\alpha)}{1+2\alpha},$$

whence:

$$u^8 = \frac{\alpha^3(2+\alpha)}{1+2\alpha} = k^2, \quad v^8 = \alpha \left(\frac{2+\alpha}{1+2\alpha}\right)^3 = \lambda^2.$$

Additionally, it is: $M = \frac{1}{1+2\alpha}$, whence having put $y = \sin T', x = \sin T$, the equation:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

becomes the following:

$$\frac{dT'}{\sqrt{(1+2\alpha)^3 - \alpha(2+\alpha)^3 \sin^2 T'}} = \frac{dT}{\sqrt{(1+2\alpha - \alpha^3(2+\alpha)\sin^2 T}},$$

or this one:

$$\frac{dT'}{\sqrt{(1+2\alpha)^3 \cos^2 T' + (1-\alpha)^3(1+\alpha)\sin^2 T'}} = \frac{dT}{\sqrt{(1+2\alpha)\cos^2 T + (1+\alpha)^3(1-\alpha)\sin^2 T}},$$

to which on get by the substitution:

$$\sin T' = \frac{(1+2\alpha)\sin T + \alpha^2 \sin^3 T}{1+\alpha(2+\alpha)\sin^2 T}.$$

1.6 A Transformation of fifth Order is propounded

15.

Now, let us treat the second simplest example in which $m = 2$,

$$V = 1 + b'x^2 + b''x^4, \quad U = x(a + a'x^2 + a''x^4), \quad A = \alpha x + \beta x^2.$$

We find:

$$AA = 1 + 2\alpha x + (2\beta + \alpha\alpha)x^2 + 2\alpha\beta x^3 + \beta\beta x^4,$$

whence:

$$AA(1+x) = 1 + x(1+2\alpha) + x^2(2\alpha+2\beta+\alpha\alpha) + x^3(2\beta+\alpha\alpha+2\alpha\beta) + x^4(2\alpha\beta+\beta\beta) + \beta\beta x^5.$$

From this we obtain:

$$\begin{aligned}
&b' = 2\alpha + 2\beta + \alpha\alpha, \quad b'' = \beta(2\alpha+\beta) \\
&a = 1 + 2\alpha, \quad a' = 2\beta + \alpha\alpha + 2\alpha\beta, \quad a'' = \beta\beta.
\end{aligned}$$

The equations $(*)$ from § 12 become:

$$a = \sqrt{\frac{k}{\lambda}} \cdot \frac{b''}{k^2}, \quad a' = \sqrt{\frac{k}{\lambda}} \cdot b', \quad a'' = \sqrt{\frac{k^5}{\lambda}}.$$

From these it follows:

$$\frac{a'a'}{aa''} = \frac{b'b'}{b''},$$

or, because one has $b' = (2+\alpha+\beta) + (\beta+\alpha\alpha)$, $a' = \beta(1+2\alpha) + (\beta+\alpha\alpha)$,

$$\frac{[(2\alpha+\beta)+(\beta+\alpha\alpha)]^2}{2\alpha+\beta} = \frac{[\beta(1+2\alpha)+(\beta+\alpha\alpha)]^2}{\beta(1+2\alpha)},$$

From this it easily follows:

$$\beta(1+2\alpha)(2\alpha+\beta) = (\beta+\alpha\alpha)^2,$$

which expanded and divided by α yields:

$$\alpha^3 = 2\beta(1+\alpha+\beta).$$

This equation can also be presented in these two ways:

$$\begin{aligned}
(\alpha\alpha+\beta)(\alpha-2\beta) &= \beta(2\alpha)(1+2\alpha) \\
(\alpha\alpha+\beta)(2-\alpha) \quad &= (\alpha-2\beta)(2\alpha+\beta),
\end{aligned}$$

whence it follows:

$$\left(\frac{2-\alpha}{\alpha-2\beta}\right)^2 = \frac{2\alpha+\beta}{\beta(1+2\alpha)}.$$

Having prepared these things the remaining are easily understood. For, we found having put $k = u^4$ and $\lambda = v^4$:

$$\frac{2\alpha + \beta}{\beta(1 + 2\alpha)} = \frac{b''}{aa''} = \frac{b'b'}{a'a'} = \frac{\lambda}{k} = \frac{v^4}{u^4},$$

whence also:

$$\frac{2 - \alpha}{\alpha - 2\beta} = \frac{v^2}{u^2}.$$

Additionally, it is $\beta = \sqrt{a'} = \sqrt[4]{\frac{k^5}{\lambda}} = \frac{u^5}{v}$, whence the equations:

$$\frac{v^4}{u^4} = \left(\frac{2 - \alpha}{\alpha - 2\beta}\right)^2 = \frac{2\alpha + \beta}{\beta(1 + 2\alpha)}, \quad \frac{2 - \alpha}{\alpha - 2\beta} = \frac{v^2}{u^2}$$

become the following:

$$\begin{aligned} 2\alpha v + u^5 &= uv^4(1 + 2\alpha) \\ u^2(2 - \alpha) &= v(v\alpha - 2u^5) \end{aligned}$$

or:

$$\begin{aligned} 2\alpha v(1 - uv^3) &= u(v^4 - u^4) \\ \alpha(v^2 + u^2) &= 2u^2(1 + u^3 v), \end{aligned}$$

whence:

$$(u^2 + v^2)(u^4 - v^4) + 4uv(1 + u^3 v)(1 - uv^3) = 0.$$

After the expansion it arises:

$$(1.) \qquad u^6 - v^6 + 5u^2 v^2(u^2 - v^2) + 4uv(1 - u^4 v^4) = 0.$$

The remaining things are found this way. From the equations:

$$\begin{aligned} 2\alpha v(1 - uv^3) &= u(v^4 - u^4) \\ \alpha(v^2 + u^2) &= 2u^2(1 + u^3 v), \end{aligned}$$

it follows:

$$\alpha = \frac{u(v^4 - u^4)}{2v(1 - uv^3)} = \frac{2u^2(1 + u^3 v)}{u^2 + v^2}.$$

From this it becomes:

$$\begin{aligned}
a = 1 + 2\alpha &= \frac{1}{v}\left(\frac{v - u^5}{1 - uv^3}\right) \\
\beta + 2\alpha &= \frac{u^5}{v} + 2\alpha = uv^2\left(\frac{v - u^5}{1 - uv^5}\right) \\
\alpha - 2\beta &= \alpha - \frac{2u^5}{v} = \frac{2u^2}{v}\left(\frac{v - u^5}{u^2 + v^2}\right) \\
2 - \alpha &= 2v\left(\frac{v - u^5}{u^2 + v^2}\right) \\
\alpha\alpha + \beta &= \frac{(\alpha - 2\beta)(2\alpha - \beta)}{2 - \alpha} = u^3\left(\frac{v - u^5}{1 - uv^3}\right).
\end{aligned}$$

Finally, one deduces from this:

$$\begin{aligned}
b' &= \beta + 2\alpha + \alpha\alpha + \beta = \frac{u(u^2 + v^2)(v - u^5)}{1 - uv^3} \\
b'' &= \frac{u^5}{v}(2\alpha + \beta) = u^6 v\left(\frac{v - u^5}{1 - uv^5}\right) \\
a &= \frac{1}{v}\left(\frac{v - u^5}{1 - uv^3}\right) \\
a' &= \frac{u^2}{v^2} \cdot b' = u^3\left(\frac{u^2 + v^2}{v^2}\right)\left(\frac{v - u^5}{1 - uv^3}\right) \\
a'' &= \frac{u^{10}}{v^2}.
\end{aligned}$$

Now, because $M = \frac{1}{a} = v\left(\frac{1 - uv^3}{v - u^5}\right)$, the transformation of fifth order will be contained in the following theorem:

Theorem

Having put:

$$(1.)\quad u^6 - v^6 + 5uuv^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

$$(2.)\quad y = \frac{v(v-u^5)x + u^3(u^2+v^2)(v-u^5)x^3 + u^{10}(1-uv^3)x^5}{v^2(1-uv^3) + uv^2(u^2+v^2)(v-u^5)x^2 + u^6v^3(v-u^5)x^4},$$

it is:

$$\frac{v(1-uv^3)dy}{\sqrt{(1-y^2)(1-v^8y^2)}} = \frac{(v-u^5)dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

1.7 How to get to Multiplication by applying a Transformation twice

16.

Considering the equations between u and v found in the propounded examples:

$$u^4 - v^4 + 2uv(1-u^2v^2) = 0$$
$$u^6 - v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-u^4v^4) = 0,$$

one cannot miss that they remain unchanged if u is put for v and $-v$ for u. From this it follows from theorem found in the first example having put:

$$u^4 - v^4 + 2uv(1-u^2v^2) = 0$$
$$y = \frac{v(v+2u^3)x + u^6x^3}{v^2 + v^3u^2(v+2u^3)x^2},$$

that it is:

$$\frac{dy}{\sqrt{(1-y^2)(1-v^8y^2)}} = \frac{v+2u^3}{v}\cdot\frac{dx}{\sqrt{(1-x^2)(1-u^8x^2)}},$$

on the other hand it is immediately derived having put:

$$z = \frac{u(u-2v^3)y + v^6y^3}{u^2 + u^3v^2(u-2v^3)y^2}$$

that it is:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8z^2)}} = \frac{u-2v^3}{u} \cdot \frac{dy}{\sqrt{(1-y^2)(1-v^8y^2)}}.$$

But it is:

$$\left(\frac{v+2u^3}{v}\right)\left(\frac{u-2v^3}{u}\right) = \frac{2(u^4-v^4)+uv(1-4u^2v^2)}{uv} = -3,$$

whence it follows:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8)}} = \frac{-3dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

To find 3 instead of -3, either z has to be changed to $-z$ or x in $-x$.

In similar manner on the one hand it is deduced from the theorem given in the second example, having put:

$$z = \frac{u(u+v^5)y - v^3(u^2+v^2)(u+v^5)y^3 + v^{10}(1+u^3v)y^5}{u^2(1+u^3v) - u^2v(u^2+v^2)(u+v^5)y^2 + u^3v^6(u+v^5)y^4},$$

that one finds:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8z^2)}} = \frac{u+v^5}{u(1+u^3v)} \cdot \frac{dy}{\sqrt{(1-y^2)(1-v^8y^2)}}.$$

Now, since from the equation:

$$u^6 - v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-uv)(1-u^4v^4) = 0$$

it follows:

$$\frac{(u+v^5)(v-u^5)}{uv(1+u^3v)(1-uv^3)} = \frac{uv(1-u^4v^4)-(u^6-v^6)}{uv(1+u^3v)(1-uv^3)} = 5,$$

we see that:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8z^2)}} = \frac{5dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

So, by means of a twice applied transformation one reaches a multiplication.

These two examples, the transformations of third and fifth order, I at first exhibited in the letters I wrote in moth of June in the year 1827 to Schuhmacher. See *Nova Astronomica* Nr. 123. And, at the same place I published the method by means of which they were found. On the other hand, they were already found by Legendre two years earlier.

1.8 On the new Notation of the elliptic Functions

17.

Having treated some algebraic questions we want to explore the analytic nature of our functions in more detail. Primarily, it is necessary to introduce a notation which will be useful in the following.

Having put $\int\limits_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = u$, the geometers used to call the angle φ the *amplitude* of the function u. Therefore, we will denote this angle by $\operatorname{ampl} u$ in the following or shorter by:

$$\varphi = \operatorname{am} u.$$

So, if

$$u = \int\limits_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

it will be:

$$x = \sin\operatorname{am} u.$$

Additionally, having put:

$$\int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = K,$$

we will call $K-u$ the complement of the function u; the amplitude of the complement we will denote by *coam* so that it is:

$$\operatorname{am}(K-u) = \operatorname{coam} u.$$

The expression $\sqrt{1-k^2\sin^2\operatorname{am} u} = \frac{d\operatorname{am} u}{du}$, following Legendre, we will denote by:

$$\Delta\operatorname{am} u = \sqrt{1-k^2\sin^2\operatorname{am} u}.$$

The complement, as it was called by Legendre, of the modulus k I will denote by k' so that:

$$kk + k'k' = 1.$$

Further, it will be from our notation:

$$K' = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k'k' \sin^2 \varphi}}.$$

The modulus which has to be kept in mind will be added either included in brackets or at the margin where it will be necessary. Not having added the modulus it is to understood that it is the same in all concerned formulas.

It pleases to call the expressions $\sin\operatorname{am} u$, $\sin\operatorname{coam} u$, $\cos\operatorname{am}$, $\cos\operatorname{coam} u$, $\Delta\operatorname{am} u$, $\Delta\operatorname{coam} u$ etc. and *trigonometric functions of the amplitude elliptic functions* in the following so that we give that name another meaning than analysts up to now. We will call u the *argument of the elliptic function* so that having put $x = \sin\operatorname{am} u$ it is $u = \arg\sin\operatorname{am} x$. From the propounded notation it will be:

$$\begin{aligned}
\sin\operatorname{coam} u &= \frac{\cos\operatorname{am} u}{\Delta\operatorname{am} u} \\
\cos\operatorname{coam} u &= \frac{k' \sin\operatorname{am} u}{\Delta\operatorname{am} u} \\
\Delta\operatorname{coam} u &= \frac{k'}{\Delta\operatorname{am} u} \\
\tan\operatorname{coam} u &= \frac{1}{k' \tan\operatorname{am} u} \\
\cot\operatorname{coam} u &= \frac{k'}{\cot\operatorname{am} u}.
\end{aligned}$$

1.9 Fundamental Formulas in the Analysis of elliptic Functions

18.

Let $\operatorname{am} u = a,\ \operatorname{am} u = b,\ \operatorname{am}(u+v) = \sigma,\ \operatorname{am}(u-v) = \vartheta$; the fundamental formulas for the addition and subtraction of elliptic functions are known:

$$\sin\sigma = \frac{\sin a\cos b\Delta b + \sin b\cos a\Delta a}{1-k^2\sin^2 a\sin^2 b}$$
$$\cos\sigma = \frac{\cos a\cos b - \sin a\sin b\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}$$
$$\Delta\sigma = \frac{\Delta a\Delta b - k^2\sin a\sin b\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}$$
$$\sin\vartheta = \frac{\sin a\cos b\Delta b - \sin b\cos a\Delta a}{1-k^2\sin^2 a\sin^2 b}$$
$$\cos\vartheta = \frac{\cos a\cos b + \sin a\sin b\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}$$
$$\Delta\vartheta = \frac{\Delta a\Delta b + k^2\sin a\sin b\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}.$$

To list all things which will be of use later we want to add the following formulas which are easily demonstrated and whose number is easily augmented:

$$(1.)\qquad \sin\sigma + \sin\vartheta = \frac{2\sin a\cos b\Delta b}{1-k^2\sin^2 a\sin^2 b}$$
$$(2.)\qquad \cos\sigma + \cos\vartheta = \frac{2\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}$$
$$(3.)\qquad \Delta\sigma + \Delta\vartheta = \frac{2\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}$$
$$(4.)\qquad \sin\sigma - \sin\vartheta = \frac{2\sin b\cos a\Delta a}{1-k^2\sin^2 a\sin^2 b}$$
$$(5.)\qquad \cos\vartheta - \cos\sigma = \frac{2\sin a\sin b\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}$$
$$(6.)\qquad \Delta\vartheta - \Delta\sigma = \frac{2k^2\sin a\cos b\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}$$
$$(7.)\qquad \sin\sigma\sin\vartheta = \frac{\sin^2 a - \sin^2 b\Delta b}{1-k^2\sin^2 a\sin^2 b}$$
$$(8.)\qquad 1+k^2\sin\sigma\sin\vartheta = \frac{\Delta^2 b + k^2\sin^2 a\cos^2 b}{1-k^2\sin^2 a\sin^2 b}$$
$$(9.)\qquad 1+\sin\sigma\sin\vartheta = \frac{\cos^2 b + \sin^2 a\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}$$

$$(10.)\qquad 1+\cos\sigma\cos\vartheta = \frac{\cos^2 a+\cos^2 b}{1-k^2\sin^2 a\sin^2 b}$$

$$(11.)\qquad 1+\Delta\sigma\Delta\vartheta = \frac{\Delta^2 a+\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}$$

$$(12.)\qquad 1-k^2\sin\sigma\sin\vartheta = \frac{\Delta^2 a+k^2\sin^2 b\cos^2 a}{1-k^2\sin^2 a\sin^2 b}$$

$$(13.)\qquad 1-\sin\sigma\sin\vartheta = \frac{\cos^2 a+\sin^2 b\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}$$

$$(14.)\qquad 1-\cos\sigma\cos\vartheta = \frac{\sin^2 a\Delta^2 b+\sin^2 b\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}$$

$$(15.)\qquad 1-\Delta\sigma\Delta\vartheta = \frac{k^2(\sin^2 a\cos^2 b+\sin^2\cos^2 a)}{1-k^2\sin^2 a\sin^2 b}$$

$$(16.)\qquad (1\pm\sin\sigma)(1\pm\sin\vartheta) = \frac{(\cos b\pm\sin a\Delta b)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(17.)\qquad (1\pm\sin\sigma)(1\mp\sin\vartheta) = \frac{(\cos a\pm\sin b\Delta a)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(18.)\qquad (1\pm\sin\sigma)(1\pm\sin\vartheta) = \frac{(\Delta b\pm k\sin a\cos b)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(19.)\qquad (1\pm\sin\sigma)(1\mp\sin\vartheta) = \frac{(\Delta a\pm k\sin b\cos a)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(20.)\qquad (1\pm\cos\sigma)(1\pm\cos\vartheta) = \frac{(\cos a\pm\cos b)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(21.)\qquad (1\pm\cos\sigma)(1\mp\cos\vartheta) = \frac{(\sin\Delta a\Delta b\mp\sin b\Delta a)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(22.)\qquad (1\pm\Delta\sigma)(1\pm\Delta\vartheta) = \frac{(\Delta a\pm\Delta b)^2}{1-k^2\sin^2 a\sin^2 b}$$

$$(23.)\qquad (1\pm\Delta\sigma)(1\mp\Delta\vartheta) = \frac{k^2\sin^2(a\mp b)}{1-k^2\sin^2 a\sin^2 b}$$

$$(24.)\qquad \sin\sigma\cos\vartheta = \frac{\sin a\cos a\Delta b+\sin b\cos b\Delta a}{1-k^2\sin^2 a\sin^2 b}$$

$$(25.)\qquad \sin\vartheta\cos\sigma = \frac{\sin a\cos a\Delta b-\sin b\cos b\Delta a}{1-k^2\sin^2 a\sin^2 b}$$

$$(26.)\qquad \sin\sigma\Delta\vartheta = \frac{\cos b\sin a\Delta a+\cos a\sin b\Delta b}{1-k^2\sin^2 a\sin^2 b}$$

$$
\begin{aligned}
(27.)\qquad & \sin\vartheta\Delta\sigma &&= \frac{\cos b\sin a\Delta a - \cos a\sin b\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
(28.)\qquad & \cos\sigma\Delta\vartheta &&= \frac{\cos a\cos b\Delta a\Delta b - k'k'\sin a\sin b}{1-k^2\sin^2 a\sin^2 b}\\
(29.)\qquad & \cos\vartheta\Delta\sigma &&= \frac{\cos a\cos b\Delta a\Delta b + k'k'\sin a\sin b}{1-k^2\sin^2 a\sin^2 b}\\
(30.)\qquad & \sin(\sigma+\vartheta) &&= \frac{2\sin a\cos a\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
(31.)\qquad & \sin(\sigma-\vartheta) &&= \frac{2\sin b\cos b\Delta a}{1-k^2\sin^2 a\sin^2 b}\\
(32.)\qquad & \cos(\sigma+\vartheta) &&= \frac{\cos^2-\sin^2 a\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}\\
(33.)\qquad & \cos(\sigma-\vartheta) &&= \frac{\cos^2 b-\sin^2\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}.
\end{aligned}
$$

1.10 On imaginary Values of elliptic Functions. The Principle of double Periodicity

19.

Let us put $\sin\varphi = i\tan\psi$ where i is written for $\sqrt{-1}$ and more used by the geometers, it will be $\cos\varphi = \sec\psi = \frac{1}{\cos\psi}$ whence $d\varphi = \frac{id\psi}{\cos\psi}$. Therefore, it is:

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{id\psi}{\sqrt{\cos^2\psi+k^2\sin^2\psi}} = \frac{id\psi}{\sqrt{1-k'k'\sin^2\psi}}.$$

We see that this from our notation becomes this equation:

$$(1.)\qquad \sin\mathrm{am}(iu,k) = i\tan\mathrm{am}(u,k').$$

From this follows:

$$
\begin{aligned}
(2.)\qquad & \cos\mathrm{am}(iu,k) &&= \sec\mathrm{am}(u,k')\\
(3.)\qquad & \tan\mathrm{am}(ik,k) &&= i\sin\mathrm{am}(u,k')\\
(4.)\qquad & \Delta\,\mathrm{am}(iu,k) &&= \frac{\Delta\,\mathrm{am}(u,k')}{\cos\mathrm{am}(u,k')} = \frac{1}{\sin\mathrm{coam}(u,k')}
\end{aligned}
$$

$$
\begin{aligned}
&(5.) \quad \sin\operatorname{coam}(iu,k) &&= \frac{1}{\Delta\operatorname{am}(u,k')}\\
&(6.) \quad \cos\operatorname{coam}(iu,k) &&= \frac{ik'}{k}\cos\operatorname{coam}(u,k')\\
&(7.) \quad \tan\operatorname{coam}(iu,k) &&= \frac{-i}{k'\sin\operatorname{am}(u,k')}\\
&(8.) \quad \Delta\operatorname{coam}(iu,k) &&= k'\sin\operatorname{coam}(u,k')
\end{aligned}
$$

Another system of formulas following from that one is this one:

$$
\begin{aligned}
&(9.) \quad \sin\operatorname{am}2iK' &&= 0\\
&(10.) \quad \sin\operatorname{am}iK' &&= \infty \quad \text{or if it pleases} \quad \pm i\infty\\
&(11.) \quad \sin\operatorname{am}(u+2iK') &&= \sin\operatorname{am}u\\
&(12.) \quad \cos\operatorname{am}(u+2iK') &&= -\cos\operatorname{am}u\\
&(13.) \quad \Delta\operatorname{am}(u+2iK') &&= -\Delta\operatorname{am}u\\
&(14.) \quad \sin\operatorname{am}(u+iK') &&= \frac{1}{k\sin\operatorname{am}u}\\
&(15.) \quad \cos\operatorname{am}(u+iK') &&= \frac{-i\Delta\operatorname{am}u}{k\sin\operatorname{am}u} = \frac{-ik'}{k\cos\operatorname{coam}u}\\
&(16.) \quad \tan\operatorname{am}(u+iK') &&= \frac{i}{\Delta\operatorname{am}u}\\
&(17.) \quad \Delta\operatorname{am}(u+iK') &&= -i\cot\sin\operatorname{am}u\\
&(18.) \quad \sin\operatorname{coam}(u+iK') &&= \frac{\Delta\operatorname{am}u}{k\cos\operatorname{am}u} = \frac{1}{k\sin\operatorname{coam}u}\\
&(19.) \quad \cos\operatorname{coam}(u+iK') &&= \frac{ik'}{k\cos\operatorname{am}u}\\
&(20.) \quad \tan\operatorname{am}(u+iK') &&= \frac{-i}{k'}\Delta\operatorname{am}u\\
&(21.) \quad \Delta\operatorname{coam}(u+iK') &&= ik'\sin\operatorname{am}u.
\end{aligned}
$$

From the preceding formulas which themselves must be considered as fundamental in the analysis of elliptic functions it becomes clear that:

a) the elliptic functions of the imaginary argument iv and the modulus k, can be transformed into other of the real argument v and modulus $k' = \sqrt{1-k^2}$. Therefore, in general it is possible to compose elliptic functions of the imaginary argument $u+iv$ and modulus k from elliptic functions of the argument

u and modulus k and other of the argument v and the modulus k'.

b) the elliptic functions enjoy double periodicity, one period being real, the other imaginary, if the modulus k is real. Both of them become imaginary, if the modulus itself is imaginary. This we will call *the principle of double periodicity*. From this, because it contains every possible periodicity, it is clear that elliptic functions must not be counted to other transcendental functions enjoying certain elegant properties, maybe more or greater than those, but they have a certain kind of perfection and the absolute.

1.11 Analytic Theory of the Transformation of elliptic Functions

20.

In the preceding paragraphs we saw that, if the polynomial functions of the element x A, B, C, D, U, V are determined in such a way that:

$$\begin{aligned} V + U &= (1 + x)AA \\ V - U &= (1 - x)BB \\ V + \lambda U &= (1 + kx)CC \\ V - \lambda U &= (1 - kx)DD, \end{aligned}$$

after having put $y = \frac{U}{V}$ that it will be:

$$\frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

M denoting a constant quantity. Now, we want the general analytical expressions of those formulas.

Let n be an arbitrary odd integer, let m and m' be arbitrary positive or negative integers which nevertheless do not have a common factor which also divides the number n, let us put:

$$\omega = \frac{mK + m'iK'}{n};$$

then, it holds:

$$
\begin{aligned}
U &= \frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}2(n-1)\omega}\right)\\
V &= (1-k^2x^2\sin^2\operatorname{am}4\omega)(1-k^2x^2\sin^2\operatorname{am}4\omega)\cdots(1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega)\\
A &= \left(1+\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1+\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1+\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\\
B &= \left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\\
C &= (1+kx\sin\operatorname{coam}4\omega)(1+kx\sin\operatorname{coam}8\omega)\cdots(1+kx\sin\operatorname{coam}2(n-1)\omega)\\
D &= (1-kx\sin\operatorname{coam}4\omega)(1-kx\sin\operatorname{coam}8\omega)\cdots(1-kx\sin\operatorname{coam}2(n-1)\omega)\\
\lambda &= k^n[\sin\operatorname{coam}4\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^4\\
M &= (-1)^{\frac{n-1}{2}}\left\{\frac{\sin\operatorname{coam}4\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}2(n-1)\omega}{\sin\operatorname{am}4\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}2(n-1)\omega}\right\}^2.
\end{aligned}
$$

Having put it like this, if $x=\sin\operatorname{am}u$, it is $y=\frac{U}{V}=\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)$.

Before we tackle the proof of the formulas itself we will indicate their transformation. For this purpose, we note the following formulas which are immediately deduced from the formulas in §. 18.

$$
\begin{aligned}
&(1.) && \sin\operatorname{am}(u+\alpha)\sin\operatorname{am}(u-\alpha) &&= \frac{\sin^2\operatorname{am}u-\sin^2\operatorname{am}\alpha}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}\\
&(2.) && \frac{[1+\sin\operatorname{am}(u+\alpha)][1+\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha} &&= \frac{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}\\
&(3.) && \frac{[1-\sin\operatorname{am}(u+\alpha)][1-\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha} &&= \frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}\\
&(4.) && \frac{[1+k\sin\operatorname{am}(u+\alpha)][1+k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha} &&= \frac{(1+k\sin\operatorname{am}u\sin\operatorname{coam}\alpha)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}\\
&(5.) && \frac{[1-k\sin\operatorname{am}(u+\alpha)][1-k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha} &&= \frac{(1-k\sin\operatorname{am}u\sin\operatorname{coam}\alpha)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}
\end{aligned}
$$

From these formulas it also follows:

$$(6.)\qquad \frac{\cos\operatorname{am}(u+\alpha)\cos\operatorname{am}(u-\alpha)}{\cos^2\operatorname{am}\alpha} = \frac{1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\alpha}}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

$$(7.)\qquad \frac{\Delta\operatorname{am}(u+\alpha)\Delta\operatorname{am}(u-\alpha)}{\Delta^2\operatorname{am}\alpha} = \frac{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{coam}\alpha}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}.$$

Having put $x = \sin\operatorname{am}u$ we obtain from formula (1.):

$$\frac{1-\frac{x^2}{\sin^2\operatorname{am}\alpha}}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{-\sin\operatorname{am}(u+\alpha)\sin\operatorname{am}(u-\alpha)}{\sin^2\operatorname{am}\alpha},$$

from the formulas (2.), (3.):

$$\frac{\left(1\pm\frac{x}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{[1\pm\sin\operatorname{am}(u+\alpha)][1\pm\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha},$$

from the formulas (4.), (5.):

$$\frac{(1\pm kx\sin\operatorname{coam}\alpha)^2}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{[1\pm k\sin\operatorname{am}(u+\alpha)][1\pm k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha}.$$

Hence, if one successively puts $4\omega,\ 8\omega,\ \cdots, 2(n-1)\omega$ for α, but $4n\omega-\alpha$ for $-\alpha$, we will obtain:

$$(8.)\qquad \frac{U}{V} = \frac{\frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}2(n-1)\omega}\right)}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{\sin\operatorname{am}u\sin\operatorname{am}(u+4\omega)\sin\operatorname{am}(u+8\omega)\cdots\sin\operatorname{am}(u+4(n-1)\omega)}{[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^2}$$

$$(9.)\qquad \frac{(1+x)AA}{V} = \frac{(1+x)\left\{\left(1+\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1+\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1+\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1+\sin\operatorname{am}u][1+\sin\operatorname{am}(u+4\omega)][1+\sin\operatorname{am}(u+8\omega)]\cdots[1+\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(10.)\qquad \frac{(1-x)BB}{V} = \frac{(1-x)\left\{\left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}4(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1-\sin\operatorname{am}u][1-\sin\operatorname{am}(u+4\omega)][1-\sin\operatorname{am}(u+8\omega)]\cdots[1-\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega][\cos\operatorname{am}8\omega]\cdots[\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(11.)\quad \frac{(1+kx)CC}{V} = \frac{(1+kx)\left\{[1+kx\sin\operatorname{coam}4\omega][1+kx\sin\operatorname{coam}8\omega]\cdots[1+kx\sin\operatorname{coam}2(n-1)\omega]\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1+k\sin\operatorname{am}u][1+k\sin\operatorname{am}(u+4\omega)][1+k\sin\operatorname{am}(u+8\omega)]\cdots[1+k\sin\operatorname{am}(u+4(n-1)\omega)]}{[\Delta\operatorname{am}4\omega\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}$$

$$(12.)\quad \frac{(1-kx)DD}{V} = \frac{(1-kx)\left\{[1-kx\sin\operatorname{coam}4\omega][1-kx\sin\operatorname{coam}8\omega]\cdots[1-kx\sin\operatorname{coam}2(n-1)\omega]\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1-k\sin\operatorname{am}u][1-k\sin\operatorname{am}(u+4\omega)][1-k\sin\operatorname{am}(u+8\omega)]\cdots[1-k\sin\operatorname{am}(u+4(n-1)\omega)]}{[\Delta\operatorname{am}4\omega\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}$$

Hence, also these formulas follow:

$$(13.)\quad \frac{\sqrt{1-x^2}AB}{V} = \sqrt{1-x^2}\,\frac{\left(1-\frac{x^2}{\sin^2\operatorname{coam}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{coam}2(n-1)\omega}\right)}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{\cos\operatorname{am}u\cos\operatorname{am}(u+4\omega)\cos\operatorname{am}(u+8\omega)\cdots\cos\operatorname{am}(u+4(n-1)\omega)}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(14.)\quad \frac{\sqrt{1-k^2x^2}CD}{V} = \sqrt{1-x^2}\,\frac{[1-k^2x^2\sin^2\operatorname{coam}4\omega][1-k^2x^2\sin^2\operatorname{coam}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{coam}2(n-1)\omega]}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{\Delta\,\mathrm{u}\,\Delta\operatorname{am}(u+4\omega)\Delta\operatorname{am}(u+8\omega)\cdots\Delta\operatorname{am}(u+4(n-1)\omega)}{[\Delta\operatorname{am}4\omega\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}.$$

1.12 Proof of the analytic Formulas for the Transformation

21.

Now, let us demonstrate that having put:

$$1-y = (1-x)\frac{\left\{\left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1-\sin\operatorname{am}u][1-\sin\operatorname{am}(u+4\omega)][1-\sin\operatorname{am}(u+8\omega)]\cdots[1-\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2},$$

that both the remaining formulas are found and this one:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}},$$

if:

$$\lambda = \qquad k^n[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^4$$
$$M = (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^2}{[\sin\operatorname{am}4\omega\sin\operatorname{am}8\omega\cdots\sin\operatorname{am}2(n-1)\omega]^2}.$$

From the propounded formula it is clear that y is not changed at all if u goes to $u+4\omega$. For, then every factor will transform into the subsequent one but the last into the first. Hence, y is generally not changed if $u+4p\omega$ is put instead of u where p denotes a negative or positive integer. On the other hand, if $u=0$, it becomes:

$$1-y=\frac{[1-\sin\operatorname{am}4\omega][1-\sin\operatorname{am}8\omega]\cdots[1-\sin\operatorname{am}4(n-1)\omega]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}=1,$$

or $y=0$. For, it easily becomes clear that it will clear:

$$\begin{aligned}-\sin\operatorname{am}4(n-1)\omega &= \operatorname{am}4\omega\\ -\sin\operatorname{am}4(n-2)\omega &= \operatorname{am}8\omega\\ \cdots\cdots\cdots\cdots&\cdots\cdots ,\end{aligned}$$

whence

$$\begin{aligned}&[1-\sin\operatorname{am}4\omega][1-\sin\operatorname{am}4(n-1)\omega]=\cos^2\operatorname{am}4\omega\\ &[1-\sin\operatorname{am}8\omega][1-\sin\operatorname{am}4(n-2)\omega]=\cos^2\operatorname{am}4\omega\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &[1-\sin\operatorname{am}2(n-1)\omega][1-\sin\operatorname{am}4(n+1)\omega]=\cos^2\operatorname{am}2(n-1)\omega\end{aligned}$$

Now, because $y=0$, if $u=0$, and y is not changed, if $u+4p\omega$ is put instead of u, y vanishes in general, u takes the following values:

$$0,\quad 4\omega,\quad 8\omega,\cdots\cdots,4(n-2)\omega,\quad 4(n-1)\omega,$$

to which the values of the quantity $x=\sin\operatorname{am}u$ correspond:

$$0,\quad \sin\operatorname{am}4\omega,\quad \sin\operatorname{am}8\omega,\cdots,\sin\operatorname{am}4(n-2)\omega,\quad \sin\operatorname{am}4(n-1)\omega$$

which can also be exhibited like this:

$$0, \quad \pm \sin \operatorname{am} 4\omega, \quad \pm \sin \operatorname{am} 8\omega, \cdots, \pm \sin \operatorname{am} 2(n-1)\omega,$$

The values of the element x which it can take while y vanishes will all be different and their number will be n. Now, from the supposed equation between x and y from which we started, having put:

$$\begin{aligned} V &= [1-k^2x^2\sin^2 \operatorname{am} 4\omega][1-k^2x^2\sin^2 \operatorname{am} 8\omega]\cdots[1-k^2x^2\sin^2 \operatorname{am} 2(n-1)\omega] \\ &= [1-k^2x^2\sin^2 \operatorname{am} 2\omega][1-k^2x^2\sin^2 \operatorname{am} 4\omega]\cdots[1-k^2x^2\sin^2 \operatorname{am}(n-1)\omega], \end{aligned}$$

and $y = \frac{U}{V}$, it becomes clear that U becomes a polynomial function of n-th order of the element x. Because this functions vanishes together with y for the following n different values of the quantity x:

$$0, \quad \pm \sin \operatorname{am} 4\omega, \quad \pm \sin \operatorname{am} 8\omega, \cdots, \pm \sin \operatorname{am} 2(n-1)\omega,$$

it necessarily takes the form:

$$\begin{aligned} U &= \frac{x}{M}\left(1-\frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1-\frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right) \\ &= \frac{x}{M}\left(1-\frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\left(1-\frac{x^2}{\sin^2 \operatorname{am} 8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \operatorname{am} 2(n-1)\omega}\right), \end{aligned}$$

where M denotes a constant. Because, having put $x = 1$, we have $1 - y = 0$ or $y = 1$, we obtain from the equation $y = \frac{U}{V}$:

$$\begin{aligned} 1 &= \frac{\left(1-\frac{1}{\sin^2 \operatorname{am} 2\omega}\right)\left(1-\frac{1}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1-\frac{1}{\sin^2 \operatorname{am}(n-1)\omega}\right)}{M[1-k^2\sin^2 \operatorname{am} 2\omega][1-k^2\sin^2 \operatorname{am} 4\omega]\cdots[1-k^2\sin^2 \operatorname{am}(n-1)2\omega]} \\ &= \frac{(-1)^{\frac{n-1}{2}}[\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^2}{M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2} \end{aligned}$$

whence

$$M = \frac{(-1)^{\frac{n-1}{2}}[\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^2}{M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2}.$$

There is a remarkable relation between the functions U, V, I mean the one mentioned above, by means of which it happens that, having put $\frac{1}{kx}$ for x, at the same time y goes to $\frac{1}{\lambda y}$ where λ denotes a constant.

For, having put $\frac{1}{kx}$ for x, the expression:

$$U = \frac{x}{M}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right)$$

becomes this expression:

$$(-)^{\frac{n-1}{2}}\frac{V}{Mx^n}\cdot\frac{1}{k^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2}.$$

On the other hand having done the same substitution,

$$V = [1 - k^2x^2 \sin^2 \operatorname{am} 2\omega][1 - k^2x^2 \sin^2 \operatorname{am} 4\omega]\cdots[1 - k^2x^2 \sin^2 \operatorname{am} 2(n-1)\omega]$$

goes over into this expression:

$$(-1)^{\frac{n-1}{2}}\frac{U}{x^n}\cdot M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am} n-1)\omega]^2.$$

Hence, having replaced x by $\frac{1}{kx}$, $y = \frac{U}{V}$ goes over into:

$$\frac{U}{V}\cdot\frac{1}{MM\cdot k^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am} n-1)\omega]^2},$$

or y into $\frac{1}{\lambda y}$ if it is put:

$$\begin{aligned}\lambda &= MMk^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^4\\ &= k^n[\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^4.\end{aligned}$$

This was to be proved.

From the propounded equation:

$$1 - y = (1-x)\frac{\left\{\left(1 - \frac{x}{\sin \operatorname{coam} 4\omega}\right)\left(1 - \frac{x}{\sin \operatorname{coam} 8\omega}\right)\cdots\left(1 - \frac{x}{\sin \operatorname{coam} 2(n-1)\omega}\right)\right\}^2}{[1 - k^2x^2 \sin^2 \operatorname{am} 4\omega][1 - k^2x^2 \sin^2 \operatorname{am} 8\omega]\cdots[1 - k^2x^2 \sin^2 \operatorname{am} 2(n-1)\omega]},$$

having put $\frac{1}{kx}$ for x, $\frac{1}{\lambda y}$ for y, what is possible from the preceding, we find:

$$\frac{1}{\lambda y} - 1 = \frac{1-kx}{\lambda U} \left\{[1 - kx \sin\operatorname{am} 4\omega][1 - kx \sin\operatorname{am} 8\omega] \cdots [1 - kx \sin\operatorname{am} 2(n-1)\omega]\right\}^2$$

which multiplied by $\lambda y = \frac{\lambda U}{V}$ yields:

$$1 - \lambda y = (1 - kx)\frac{\left\{[1 - kx \sin\operatorname{am} 4\omega][1 - kx \sin\operatorname{am} 8\omega] \cdots [1 - kx \sin\operatorname{am} 2(n-1)\omega]\right\}^2}{V}.$$

Furthermore, it is clear that $y = \frac{U}{V}$ goes over into $-y$ if x is changed to $-x$ having done which we therefore immediately also obtain $1 + y$, $1 + \lambda y$ from $1 - y$, $1 - \lambda y$.

Therefore, we have now found polynomial functions U, V of the element x of such a kind that it is

$$\begin{aligned}
V + U &= V(1+y) &&= (1+x)AA \\
V - U &= V(1-y) &&= (1-x)BB \\
V + \lambda U &= V(1+\lambda y) &&= (1+kx)CC \\
V - \lambda U &= V(1-\lambda y) &&= (1-kx)DD,
\end{aligned}$$

where A, B, C, D themselves also denote polynomial functions of the element x. But, from this according to the initially proved principles of the transformation it immediately follows:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2}}.$$

We obtain the multiplicator M as we will call it from the observation in § 14. Hence, now all general analytical formulas concerning the theory of the transformation of elliptic functions are demonstrated.

22.

The propounded proof is found from the one we gave in the *Nova Astronomica* Nr. 127 edited by Schuhmacher where ω is put instead of $\frac{K}{n}$, $(-1)^{\frac{n-1}{2}}$ instead of M, while all other quantities remain unchanged. First, I had communicated the general analytic theorem on the transformation in a slightly different form at the same place with the analysts in Nr. 123. Legendre, the greatest judge in this doctrine, wanted to review that proof ibidem in Nr. 130 in great detail. This in multiple ways venerable man observed that there that the equation:

$$V\frac{dU}{dx} - U\frac{dV}{dx} = \frac{ABCD}{M} = \frac{T}{M},$$

by means of which the proof is given and which in this treatise followed from principles of a mere algebraic transformation can be proved also without those analytically. Because from this remark of this remarkable man much light is shed on our theorem let us demonstrate that equation with in the same way as Legendre using less.

The propounded equation:

$$V\frac{dU}{dx} - U\frac{dV}{dx} = \frac{ABCD}{M} = \frac{T}{M}$$

can also exhibited this way:

$$\frac{dU}{Udx} - \frac{dV}{Vdx} = \frac{d\log U}{dx} - \frac{d\log V}{dx} = \frac{ABCD}{MUV} = \frac{T}{MUV}.$$

But we found:

$$\begin{aligned} U &= \frac{x}{M}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right) \\ V &= [1 - k^2x^2\sin^2 \operatorname{am} 2\omega][1 - k^2x^2\sin^2 \operatorname{am} 4\omega]\cdots[1 - k^2x^2\sin^2 \operatorname{am}(n-1)\omega], \end{aligned}$$

whence:

$$\frac{d\log U}{dx} - \frac{d\log V}{dx} = \frac{1}{x} + \sum\left\{\frac{-2x}{\sin^2 \operatorname{am} 2q\omega} + \frac{2k^2x \operatorname{am} 2q\omega}{1 - k^2x^2\sin^2 \operatorname{am} 2q\omega}\right\},$$

after having assigned the values $1, 2, 3, \cdots, \frac{n-1}{2}$ to the number denoted by q. Furthermore, we found:

$$\begin{aligned} AB &= \left(1 - \frac{x^2}{\sin^2 \operatorname{coam} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{coam} 4\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2 \operatorname{coam}(n-1)\omega}\right) \\ CD &= [1 - k^2x^2\sin^2 \operatorname{coam} 2\omega][1 - k^2x^2\sin^2 \operatorname{coam} 4\omega]\cdots[1 - k^2x^2\sin^2 \operatorname{coam}(n-1)\omega], \end{aligned}$$

whence

$$\frac{T}{MUV} = \frac{ABCD}{MUV} = \frac{x \prod \left(1 - \frac{x^2}{\sin^2 \operatorname{coam}} 2p\omega\right) (1 - k^2 x^2 \sin^2 \operatorname{coam} 2p\omega)}{x^2 \prod \left(1 - \frac{x^2}{\sin^2 \operatorname{am}} 2p\omega\right) (1 - k^2 x^2 \sin^2 \operatorname{am} 2p\omega)},$$

if in the products, for the sake of brevity denoted by the prefixed sign $\prod$, the values $1, 2, 3, \cdots \frac{n-1}{2}$ are assigned to the element p. This expression can be decomposed into simple fractions such that it takes this form:

$$\frac{1}{x} + \sum \left(\frac{A^{(q)} x}{\sin^2 \operatorname{am} 2q\omega} + \frac{B^{(q)}}{1 - k^2 x^2 \sin^2 \operatorname{am} 2q\omega} \right);$$

Having done this, in order to reach that which was propounded it must be demonstrated that it will be:

$$A^{(q)} = -1, \quad B^{(q)} = 2k^2 \sin^2 \operatorname{am} 2q\omega$$

In the following we will denote by the prefixed sign $\prod^{(q)}$ the product formed in such a way that the values $1, 2, 3, \cdots, \frac{n-1}{2}$ are assigned to the element p, but the value $p = q$ is omitted. Hence, from the well-known theories of simple fractions it follows:

$$A^{(q)} = (1 - k^2 \sin^2 \operatorname{am} q\omega \sin^2 \operatorname{coam} 2q\omega) \frac{\prod \left(\frac{1 - \frac{\sin^2 \operatorname{am} 2q\omega}{\sin^2 \operatorname{coam} 2p\omega}}{1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{am} 2p\omega} \right)}{\prod^{(q)} \left(\frac{1 - \frac{\sin^2 \operatorname{am} 2q\omega}{\sin^2 \operatorname{am} 2p\omega}}{1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2p\omega} \right)}.$$

Now, it is from the formulas we exhibited above:

$$\frac{1 - \frac{\sin^2 \operatorname{am} 2q\omega}{\sin^2 \operatorname{coam} 2p\omega}}{1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{am} 2p\omega} = \frac{\cos \operatorname{am}(2q + 2p)\omega \cos \operatorname{am}(2p - 2q)\omega}{\cos^2 \operatorname{am} 2p\omega}$$

$$\frac{1 - \frac{\sin^2 \operatorname{am} 2q\omega}{\sin^2 \operatorname{am} 2p\omega}}{1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2p\omega} = \frac{\cos \operatorname{coam}(2q + 2p)\omega \cos \operatorname{coam}(2p - 2q)\omega}{\cos^2 \operatorname{coam} 2p\omega}.$$

But, it easily becomes clear, after having removed the factors which are the same in the denominator and numerator, that it is:

$$\prod \frac{\cos \operatorname{am}(2q+2p)\omega \cos \operatorname{am}(2q-2p)\omega}{\cos^2 \operatorname{am} 2p\omega} = \frac{\pm 1}{\cos \operatorname{am} 2q\omega}$$

$$\prod^{(q)} \frac{\cos \operatorname{am}(2q+2p)\omega \cos \operatorname{am}(2q-2p)\omega}{\cos^2 \operatorname{am} 2p\omega} = \frac{\mp 1}{\cos \operatorname{am} 2q\omega} \cdot \frac{\cos^2 \operatorname{coam} 2q\omega}{\cos \operatorname{coam} 4p\omega} = \frac{\mp \cos \operatorname{coam} 2q\omega}{\cos \operatorname{coam} 4q\omega},$$

whence:

$$A^{(q)} = \frac{-(1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2q\omega) \cos \operatorname{coam} 4q\omega}{\cos \operatorname{am} 2q\omega \cos \operatorname{coam} 2q\omega}.$$

But, from the noted formula on the duplication it is:

$$\begin{aligned} \cos \operatorname{coam} 4q\omega &= \frac{2k' \sin \operatorname{am} 2q\omega \cos \operatorname{am} 2q\omega \Delta \operatorname{am} 2q\omega}{1 - 2k^2 \sin^2 \operatorname{am} 2q\omega + k^2 \sin^4 \operatorname{am} 2q\omega} \\ &= \frac{2k' \sin \operatorname{am} 2q\omega \cos \operatorname{am} 2q\omega \Delta \operatorname{am} 2q\omega}{\Delta^2 \operatorname{am} 2q\omega - k^2 \sin^2 2q\omega \cos^2 \operatorname{am} 2q\omega} \\ &= \frac{2 \cos \operatorname{coam} 2q\omega \cos \operatorname{coam} 2q\omega}{1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2q\omega}, \end{aligned}$$

whence finally, as it was to be demonstrated, $A^{(q)} = -2$. In completely the same manner the other equation: $B^{(q)} = 2k^2 \sin^2 \operatorname{am} 2q\omega$ can be proved; this is nevertheless, already having found $A^{(q)} = -2$, easier archived the following way.

It easily becomes clear, having put $\frac{1}{kx}$ instead of x that the following expression is not changed:

$$\prod \frac{\left(1 - \frac{x^2}{\sin^2 \operatorname{coam} 2p\omega}\right)(1 - k^2 x^2 \sin^2 \operatorname{coam} 2p\omega)}{(1 - k^2 x^2 \sin^2 \operatorname{am} 2p\omega)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2p\omega}\right)},$$

which we see to be able to be put equal to the expression:

$$1 + \sum \frac{-2x^2}{\sin^2 \operatorname{am} 2q\omega - x^2} + \sum \frac{B^{(q)} x^2}{1 - k^2 x^2 \sin^2 \operatorname{am} 2q\omega}.$$

But this expression, having put $\frac{1}{kx}$ instead of x, goes over into this one:

$$1 + \sum \frac{2}{1 - k^2x^2 \sin^2 \operatorname{am} 2q\omega} + \sum \frac{-B^{(q)}}{k^2(\sin^2 \operatorname{am} 2q\omega - x^2)}$$

$$= 1 + \sum \left(2 - \frac{B^{(q)}}{k^2 \sin^2 \operatorname{am} 2q\omega}\right) + \sum \frac{2k^2x^2 \sin^2 \operatorname{am} 2q\omega}{1 - k^2x^2 \sin^2 \operatorname{am} 2q\omega} + \sum \frac{-B^{(q)}}{k^2 \sin^2 \operatorname{am} 2q\omega} \cdots \frac{x^2}{\sin^2 \operatorname{am} 2q\omega - x^2},$$

whence that this expression stays unchanged which must happen it has to be:

$$B^{(q)} = 2k^2 \sin^2 \operatorname{am} 2q\omega.$$

Q.D.E.

23.

From formula (14.) in §20 it follows:

$$\sqrt{1-\lambda^2y^2} = \sqrt{1-k^2x^2}\frac{CD}{V}$$
$$= \sqrt{1-k^2x^2}\frac{[1-k^2x^2\sin^2\operatorname{coam}2\omega][1-k^2x^2\sin^2\operatorname{coam}4\omega]\cdots[1-k^2x^2\sin^2\operatorname{coam}(n-1)\omega]}{[1-k^2x^2\sin^2\operatorname{am}2\omega][1-k^2x^2\sin^2\operatorname{am}4\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}(n-1)\omega]}.$$

Having put $x = 1$, whence also $y = 1$ and $\sqrt{1-\lambda^2} = \lambda'$, it is:

$$\lambda' = k' \left\{\frac{\Delta \operatorname{coam} 2\omega \Delta \operatorname{coam} 4\omega \cdots \Delta \operatorname{coam}(n-1)\omega}{\Delta \operatorname{am} 2\omega \Delta \operatorname{am} 4\omega \cdots \Delta \operatorname{am}(n-1)\omega}\right\}^2$$

But, it is:

$$\Delta\Delta \operatorname{coam} = \frac{k'}{\operatorname{am}},$$

whence:

$$(1.) \qquad \lambda' = \frac{k'^n}{[\Delta \operatorname{am} 2\omega \Delta \operatorname{am} 4\omega \cdots \Delta \operatorname{am}(n-1)\omega]^4}$$

Further, using the formulas:

$$(2.) \qquad \lambda = k^n[\sin\operatorname{coam}2\omega \sin\operatorname{coam}4\omega \cdots \sin\operatorname{coam}(n-1)\omega]^4$$

$$(3.) \qquad M = (-1)^{\frac{n-1}{2}} \frac{[\sin\operatorname{coam}2\omega \sin\operatorname{coam}4\omega \cdots \sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega \sin\operatorname{am}4\omega \cdots \sin\operatorname{am}(n-1)\omega]^2},$$

we obtain:

$$\begin{aligned}
&(4.) \quad \frac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\frac{\lambda}{k^n}} &&= [\sin\operatorname{am} 2\omega \sin\operatorname{am} 4\omega \cdots \sin\operatorname{am}(n-1)\omega]^2\\
&(5.) \quad \sqrt{\frac{\lambda k'^n}{\lambda' k^n}} &&= [\cos\operatorname{am} 2\omega \cos\operatorname{am} 4\omega \cdots \cos\operatorname{am}(n-1)\omega]^2\\
&(6.) \quad \sqrt{\frac{k'^n}{\lambda'}} &&= [\Delta\operatorname{am} 2\omega \Delta\operatorname{am} 4\omega \cdots \Delta\operatorname{am}(n-1)\omega]^2\\
&(7.) \quad \frac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\frac{\lambda'}{k'^n}} &&= [\tan\operatorname{am} 2\omega \tan\operatorname{am} 4\omega \cdots \tan\operatorname{am}(n-1)\omega]^2\\
&(8.) \quad \sqrt{\frac{\lambda}{k^n}} &&= [\sin\operatorname{coam} 2\omega \sin\operatorname{coam} 4\omega \cdots \sin\operatorname{coam}(n-1)\omega]^2\\
&(9.) \quad \frac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\frac{\lambda\lambda' k'^{n-2}}{k^n}} &&= [\cos\operatorname{coam} 2\omega \cos\operatorname{coam} 4\omega \cdots \cos\operatorname{coam}(n-1)\omega]^2\\
&(10.) \quad \sqrt{\lambda' k'^{n-2}} &&= [\Delta\operatorname{coam} 2\omega \Delta\operatorname{coam} 4\omega \cdots \Delta\operatorname{coam}(n-1)\omega]^2\\
&(11.) \quad (-1)^{\frac{n-1}{2}} M\sqrt{\frac{1}{\lambda' k'^{n-2}}} &&= [\tan\operatorname{coam} 2\omega \tan\operatorname{coam} 4\omega \cdots \tan\operatorname{coam}(n-1)\omega]^2.
\end{aligned}$$

By means of these formulas the formulas (8.), (13.), (14.) §20 go over into the following

$$\begin{aligned}
&(12.) \quad \sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k^n}{\lambda}} \sin\operatorname{am} u \sin\operatorname{am}(u+4\omega) \sin\operatorname{am}(u+8\omega) \cdots \sin\operatorname{am}(u+4(n-1)\omega)\\
&(13.) \quad \cos\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{\lambda' k^n}{\lambda k'^n}} \cos\operatorname{am} u \cos\operatorname{am}(u+4\omega) \cos\operatorname{am}(u+8\omega) \cdots \cos\operatorname{am}(u+4(n-1)\omega)\\
&(14.) \quad \Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{\lambda'}{k'^n}} \Delta\operatorname{am} u \Delta\operatorname{am}(u+4\omega) \Delta\operatorname{am}(u+8\omega) \cdots \Delta\operatorname{am}(u+4(n-1)\omega)
\end{aligned}$$

whence also:

$$(15.) \quad \tan\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k'^n}{\lambda'}} \tan\operatorname{am} u \tan\operatorname{am}(u+4\omega) \tan\operatorname{am}(u+8\omega) \cdots \tan\operatorname{am}(u+4(n-1)\omega)$$

So, another system of formulas is found. From equation (4.) it follows:

$$\frac{\lambda}{M^2k^n} = [\sin\operatorname{am} 2\omega \sin\operatorname{am} 4\omega \cdots \sin\operatorname{am}(n-1)\omega]^4,$$

whence:

$$y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \frac{x}{M}\prod \frac{1 - \frac{x^2}{\sin^2 \sin\operatorname{am} 2p\omega}}{1 - k^2x^2\sin^2\operatorname{am} 2p\omega} = \frac{kM}{\lambda} x \prod \frac{x^2 - \sin^2\operatorname{am} 2p\omega}{x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}},$$

or:

$$0 = x\prod(x^2 - \sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)\prod\left(x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right).$$

The roots of this equation of n-th order are:

$$x = \sin\operatorname{am} u, \quad \sin\operatorname{am}(u+4\omega), \quad \sin\operatorname{am}(u+8\omega), \cdots, \sin\operatorname{am}(u+4(n-1)\omega),$$

whence we obtain the identity:

$$x\prod(x^2 - \sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)\prod\left(x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right)$$

$$= [x - \sin^2\operatorname{am} u][x - \sin^2\operatorname{am}(u+4\omega)][x - \sin^2\operatorname{am}(u+8\omega)]\cdots[x - \sin^2\operatorname{am}(u+4(n-1)\omega)].$$

From this the sum of roots arise:

$$(16.) \quad \sum \sin(u + 4q\omega) = \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right).$$

In the same way it is found

$$(17.) \quad \sum \cos\operatorname{am}(u + 4q\omega) = \frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

$$(18.) \quad \sum \Delta\operatorname{am}(u + 4q\omega) = \frac{(-1)^{\frac{n-1}{2}}\lambda}{M}\Delta\operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

$$(19.) \quad \sum \tan\operatorname{am}(u + 4q\omega) = \frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M}, \lambda\right),$$

in which formula the values $0, 1, 2, 3, \cdots, n-1$ are assigned to the number q. It is convenient to represent also these formulas in this way:

$$\begin{aligned}
\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sin\operatorname{am} + \sum[\sin\operatorname{am}(u+4q\omega)+\sin\operatorname{am}(u-4q\omega)]\\
\frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \cos\operatorname{am} + \sum[\cos\operatorname{am}(u+4q\omega)+\cos\operatorname{am}(u-4q\omega)]\\
\frac{(-1)^{\frac{n-1}{2}}\lambda}{M}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \Delta\operatorname{am} \quad + \sum[\Delta\operatorname{am}(u+4q\omega)+\Delta\operatorname{am}(u-4q\omega)]\\
\frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \tan\operatorname{am} + \sum[\tan\operatorname{am}(u+4q\omega)+\tan\operatorname{am}(u-4q\omega)],
\end{aligned}$$

where the number q takes the values $1, 2, 3, \cdots, \frac{n-1}{2}$. Now, let the following formulas be noted:

$$\begin{aligned}
\sin\operatorname{am}(u+4q\omega)+\sin\operatorname{am}(u-4q\omega) &= \frac{2\cos\operatorname{am}4q\omega\Delta\operatorname{am}4q\omega\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
\cos\operatorname{am}(u+4q\omega)+\cos\operatorname{am}(u-4q\omega) &= \frac{2\cos\operatorname{am}4q\omega\cos\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
\Delta\operatorname{am}(u+4q\omega) \quad +\Delta\operatorname{am}(u-4q\omega) \quad &= \frac{2\Delta\operatorname{am}4q\omega\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
\tan\operatorname{am}(u+4q\omega)+\tan\operatorname{am}(u-4q\omega) &= \frac{2\Delta\operatorname{am}4q\omega\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}4q\omega-\Delta^2\operatorname{am}4q\omega\sin^2\operatorname{am}u},
\end{aligned}$$

by means of which the formulas (16.) – (19.) go over into these:

$$\begin{aligned}
&(20.) \quad \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sin\operatorname{am}u + \sum\frac{2\cos\operatorname{am}4q\omega\Delta\operatorname{am}4q\omega\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
&(21.) \quad \frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) = \cos\operatorname{am}u + \sum\frac{2\cos\operatorname{am}4q\omega\cos\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
&(22.) \quad \frac{(-1)^{\frac{n-1}{2}}\lambda}{M}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) = \Delta\operatorname{am} \quad + \sum\frac{2\Delta\operatorname{am}4q\omega\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
&(23.) \quad \frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) = \tan\operatorname{am}u + \sum\frac{2\Delta\operatorname{am}4q\omega\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}4q\omega-\Delta^2\operatorname{am}4q\omega\sin^2\operatorname{am}u},
\end{aligned}$$

which are also obtained, where the formulas propounded above are resolved into simple fractions from known methods.

1.13 On various Transformations of the same Order. Two real Transformations, of a larger Modulus into a smaller and of smaller into a larger

24.

We saw that we can assign to the element ω an arbitrary value of the form $\frac{mK+m'iK}{n}$ m and m' denoting positive or negative integer numbers which nevertheless, if n is a composite number, do not have a common factor of n. But it easily becomes clear, if q is a prime number, that the values $\frac{qmK+im'iK'}{n}$ will not exhibit different substitutions. Hence, if n itself is a prime number, all values of the element ω which yield different transformations will be:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K+iK'}{n}, \quad \frac{K+2iK'}{n}, \quad \frac{K+3iK'}{n}, \quad \cdots \quad \frac{K+(n-1)iK'}{n},$$

or also:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K+iK'}{n}, \quad \frac{2K+iK'}{n}, \quad \frac{3K+iK'}{n}, \dots, \frac{(n-1)K+iK'}{n},$$

or, if it pleases:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K \pm iK'}{n}, \quad \frac{K \pm 2iK'}{n}, \quad \frac{K \pm 3iK'}{n}, \dots, \frac{K \pm \frac{n-1}{2}iK'}{n},$$

or also:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K \pm iK'}{n}, \quad \frac{2K \pm iK'}{n}, \quad \frac{3K \pm iK'}{n}, \dots, \frac{\frac{n-1}{2}K \pm iK'}{n},$$

whose number is $n+1$. And indeed we saw that in the transformations of third and fifth order propounded above as examples that the equations between $u = \sqrt[4]{k}$ and $\sqrt[4]{v}$ which we called *modular equations* raised to fourth and sixth degree, respectively. But if n is a composite number this number is vastly augmented; for, the cases in which either m or m' or even both have a certain common factor but the common factor of m, m' is not a factor of n, accrue. In general, the following theorem holds:

"The number of mutually different substitutions of n-th order by means of which it is possible to transform elliptic functions is equal to the sum of factors of n which number nevertheless, if n is not square-free, contains substitutions mixed from a transformation and multiplication, and hence, if n is a perfect square, contains the multiplication itself."

Therefore, this sum of factors will denote the grade to which for given number n the modular equation will raise, where it is to be noted, if n is a square, that one of the total number of roots will yield $k = \lambda$, and generally that, if $n = mv^2$ m^2 denotes a mixed square dividing n, of the total number of roots also all will be roots of the modular equation which belongs to v itself.

Among the values of the element ω propounded above which in the case in which n is prime which case, since the remaining reduce to it, is convenient to consider it separately, yielded the total amount of transformations, generally speaking only two are found which yield real transformations; namely $\omega = \frac{K}{n}$, $\omega' = \frac{iK'}{n}$. In the following, we will call the latter the *first* transformation, the other the *second*. And the moduli corresponding to them we will denote respectively by λ, λ_1 and their complements by λ', λ'_1. The arguments of the amplitude $\frac{\pi}{2}$ corresponding to these moduli (Legendre calls them complete elliptic integrals) we will denote by Λ, Λ_1, Λ', Λ'_1. Our general formulas for these two case are the following.

I.

Formulas for the first real Transformations of the Modulus k into the Modulus λ.

$$\lambda = k^n \left\{ \sin\operatorname{coam}\frac{2K}{n}\sin\operatorname{coam}\frac{4K}{n}\cdots\sin\operatorname{coam}\frac{(n-1)K}{n}\right\}^4$$

$$\lambda' = \frac{k'^n}{\left\{\Delta\operatorname{coam}\frac{2K}{n}\Delta\operatorname{coam}\frac{4K}{n}\cdots\Delta\operatorname{coam}\frac{(n-1)K}{n}\right\}^4}$$

$$M = \left\{\frac{\sin\operatorname{coam}\frac{2K}{n}\sin\operatorname{coam}\frac{4K}{n}\cdots\sin\operatorname{coam}\frac{(n-1)K}{n}}{\sin\operatorname{am}\frac{2K}{n}\sin\operatorname{am}\frac{4K}{n}\cdots\sin\operatorname{am}\frac{(n-1)K}{n}}\right\}^2$$

$$\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \frac{\frac{\sin\operatorname{am}u}{M}\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{am}\frac{2K}{n}}\right)\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1-k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)}$$

$$= (-1)^{\frac{n-1}{2}}\sqrt{\frac{k^n}{\lambda}}\sin\operatorname{am}u\sin\operatorname{am}\left(u+\frac{4K}{n}\right)\sin\operatorname{am}\left(u+\frac{8K}{n}\right)\cdots\sin\operatorname{am}\left(u+\frac{4(n-1)K}{n}\right)$$

$$\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) = \frac{\cos\operatorname{am}u\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\frac{2K}{n}}\right)\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\frac{4K}{n}}\right)\cdots\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\frac{(n-1)K}{n}}\right)}{\left(1-k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)}$$

$$= \sqrt{\frac{\lambda' k^n}{\lambda k'^n}}\cos\operatorname{am}u\cos\operatorname{am}\left(u+\frac{4K}{n}\right)\cos\operatorname{am}\left(u+\frac{8K}{n}\right)\cdots\cos\operatorname{am}\left(u+\frac{4(n-1)K}{n}\right)$$

$$\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) = \frac{\Delta\operatorname{am}\left(1-k^2\sin^2\operatorname{coam}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{coam}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{coam}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)}{\left(1-k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)}$$

$$= \sqrt{\frac{\lambda'}{k'^n}}\Delta\operatorname{am}u\Delta\operatorname{am}\left(u+\frac{4K}{n}\right)\Delta\operatorname{am}\left(u+\frac{8K}{n}\right)\cdots\Delta\operatorname{am}\left(u+\frac{4(n-1)K}{n}\right)$$

$$\sqrt{\frac{1\mp\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}{1\pm\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}}$$

$$= \sqrt{\frac{1-\sin\operatorname{am}u}{1+\sin\operatorname{am}u}}\cdot\frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4K}{n}}\right)\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{8K}{n}}\right)\cdots\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2(n-1)K}{n}}\right)}{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4K}{n}}\right)\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{8K}{n}}\right)\cdots\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2(n-1)K}{n}}\right)}$$

$$\sqrt{\frac{1\mp\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}{1\pm\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}}$$

$$= \sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}}\cdot\frac{\left(1-k\sin\operatorname{coam}\frac{4K}{n}\sin\operatorname{am}u\right)\left(1-k\sin\operatorname{coam}\frac{8K}{n}\sin\operatorname{am}u\right)\cdots\left(1-k\sin\operatorname{coam}\frac{2(n-1)K}{n}\sin\operatorname{am}u\right)}{\left(1+k\sin\operatorname{coam}\frac{4K}{n}\sin\operatorname{am}u\right)\left(1+k\sin\operatorname{coam}\frac{8K}{n}\sin\operatorname{am}u\right)\cdots\left(1+k\sin\operatorname{coam}\frac{2(n-1)K}{n}\sin\operatorname{am}u\right)}$$

$$
\begin{aligned}
\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sin\operatorname{am}u + 2\sum\frac{(-1)^q\cos\operatorname{am}\frac{2qK}{n}\Delta\operatorname{am}\frac{2qK}{n}\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}\\
\frac{\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \cos\operatorname{am}u + 2\sum\frac{(-1)^q\cos\operatorname{am}\frac{2qK}{n}\cos\operatorname{am}u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}\\
\frac{1}{M}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \Delta\operatorname{am}u + 2\sum\frac{\Delta\operatorname{am}\frac{2qK}{n}\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}\\
\frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \tan\operatorname{am}u + 2\sum\frac{\tan\operatorname{am}\frac{2qK}{n}\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}\frac{2qK}{n}-\Delta^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}.
\end{aligned}
$$

II.

A. Formulas for the second real Transformation, of the Modulus k into the Modulus λ_1 under an imaginary Form

$$\lambda_1 = k^n \left\{ \sin\operatorname{coam} \frac{2iK'}{n} \sin\operatorname{coam} \frac{4iK'}{n} \cdots \sin\operatorname{coam} \frac{(n-1)K}{n} \right\}^4$$

$$\lambda_1' = \frac{k'^n}{\left\{ \Delta\operatorname{coam} \frac{2iK'}{n} \Delta\operatorname{coam} \frac{4iK'}{n} \cdots \Delta\operatorname{coam} \frac{(n-1)iK'}{n} \right\}^4}$$

$$M = (-1)^{\frac{n-1}{2}} \left\{ \frac{\sin\operatorname{coam} \frac{2iK'}{n} \sin\operatorname{coam} \frac{4iK'}{n} \cdots \sin\operatorname{coam} \frac{(n-1)iK'}{n}}{\sin\operatorname{am} \frac{2iK'}{n} \sin\operatorname{am} \frac{4iK'}{n} \cdots \sin\operatorname{am} \frac{(n-1)iK'}{n}} \right\}^2$$

$$\sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = \frac{\frac{\sin\operatorname{am} u}{M_1}\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{2iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-1)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{k^n}{\lambda_1}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4iK'}{n}\right) \sin\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\cos\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = \frac{\cos\operatorname{am} u\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{2iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{(n-1)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda_1' k^n}{\lambda_1 k'^n}} \cos\operatorname{am} u \cos\operatorname{am}\left(u + \frac{4iK'}{n}\right) \cos\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \cos\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\Delta\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = \frac{\Delta\operatorname{am} u\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{(n-2)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda_1'}{k'^n}} \Delta\operatorname{am} u \Delta\operatorname{am}\left(u + \frac{4iK'}{n}\right) \Delta\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \Delta\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\sqrt{\frac{1 - \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}{1 + \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}}$$

$$= \sqrt{\frac{1 - \sin\operatorname{am} u}{1 + \sin\operatorname{am} u}} \cdot \frac{\left(1 - \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{2iK'}{n}}\right)\left(1 - \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{(n-1)iK'}{n}}\right)}{\left(1 + \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{2iK'}{n}}\right)\left(1 + \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{4iK'}{n}}\right)\cdots\left(1 + \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{(n-1)iK'}{n}}\right)}$$

$$\sqrt{\frac{1 - \lambda_1 \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}{1 + \lambda_1 \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}}$$

$$= \sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}} \cdot \frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{iK'}{n}}\right)\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{3iK'}{n}}\right)\cdots\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{(n-2)iK'}{n}}\right)}{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{iK'}{n}}\right)\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{3iK'}{n}}\right)\cdots\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{(n-2)iK'}{n}}\right)}$$

$$\begin{aligned}
\frac{\lambda_1}{kM_1}\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) &= \sin\operatorname{am}u + \frac{2}{k}\sum\frac{\cos\operatorname{am}\frac{(2q-1)iK'}{n}\Delta\operatorname{am}\frac{(2q-1)iK'}{n}\sin\operatorname{am}u}{\sin^2\operatorname{am}\frac{(2q-1)iK'}{n}-\sin^2\operatorname{am}u}\\
\frac{(-1)^{\frac{n-1}{2}}\lambda_1}{kM_1}\cos\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) &= \cos\operatorname{am}u + \frac{2(-1)^{\frac{n-1}{2}}}{ik}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)iK'}{n}\Delta\operatorname{am}\frac{(2q-1)iK'}{n}\cos\operatorname{am}u}{\sin^2\operatorname{am}\frac{(2q-1)iK'}{n}-\sin^2\operatorname{am}u}\\
\frac{(-1)^{\frac{n-1}{2}}}{M_1}\Delta\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) &= \Delta\operatorname{am}u \quad + \frac{2(-1)^{\frac{n-1}{2}}}{i}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)iK'}{n}\Delta\operatorname{am}\frac{(2q-1)iK'}{n}\cos\operatorname{am}u}{\sin^2\operatorname{am}\frac{(2q-1)iK'}{n}-\sin^2\operatorname{am}u}\\
\frac{\lambda_1'}{k'M_1}\tan\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) &= \tan\operatorname{am}u + 2\sum\frac{(-1)^q\Delta\operatorname{am}\frac{2qiK'}{n}\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}\frac{2qiK'}{n}-\Delta^2\operatorname{am}\frac{2qiK'}{n}\sin^2\operatorname{am}u}
\end{aligned}$$

II.

B. Formulas for the second real Transformation in real Form

$$\lambda_1 = \frac{k^n}{\left\{\Delta\,\mathrm{am}\left(\frac{2K'}{n},k'\right)\left(\frac{4K'}{n},k'\right)\cdots\left(\frac{(n-1)K'}{n},k'\right)\right\}^4}$$

$$\lambda_1' = k'^n\left\{\sin\mathrm{coam}\left(\frac{2K'}{n},k'\right)\sin\mathrm{coam}\left(\frac{4K'}{n},k'\right)\cdots\sin\mathrm{coam}\left(\frac{(n-1)K'}{n},k'\right)\right\}^4$$

$$M_1 = \left\{\frac{\sin\mathrm{coam}\left(\frac{2K'}{n},k'\right)\sin\mathrm{coam}\left(\frac{4K'}{n},k'\right)\sin\mathrm{coam}\cdots\left(\frac{(n-1)K'}{n},k'\right)}{\sin\mathrm{am}\left(\frac{2K'}{n},k'\right)\sin\mathrm{am}\left(\frac{4K'}{n},k'\right)\sin\mathrm{am}\cdots\left(\frac{(n-1)K'}{n},k'\right)}\right\}^2$$

$$\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \frac{\frac{\sin\mathrm{am}\,u}{M_1}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{2K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{4K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{(n-1)K'}{n},k'\right)}\right\}}{\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{3K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)}\right\}}$$

$$\cos\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \frac{\cos\mathrm{am}\,u\left\{1-\sin^2\mathrm{am}\,u\Delta^2\,\mathrm{am}\left(\frac{2K'}{n}\right)\right\}\left\{1-\sin^2\mathrm{am}\,u\Delta^2\,\mathrm{am}\left(\frac{4K'}{n}\right)\right\}\left\{1-\sin^2\mathrm{am}\,u\Delta^2\,\mathrm{am}\left(\frac{(n-1)K'}{n}\right)\right\}}{\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{3K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)}\right\}}$$

$$\Delta\,\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \frac{\Delta\,\mathrm{am}\,u\left\{1-\sin^2\mathrm{am}\,u\Delta^2\,\mathrm{am}\left(\frac{K'}{n}\right)\right\}\left\{1-\sin^2\mathrm{am}\,u\Delta^2\,\mathrm{am}\left(\frac{3K'}{n}\right)\right\}\left\{1-\sin^2\mathrm{am}\,u\Delta^2\,\mathrm{am}\left(\frac{(n-2)K'}{n}\right)\right\}}{\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{3K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\tan^2\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)}\right\}}$$

$$\sqrt{\frac{1-\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)}{1+\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)}}$$

$$= \sqrt{\frac{1-\sin\mathrm{am}\,u}{1+\sin\mathrm{am}\,u}}\cdot\frac{\left\{1-\sin\mathrm{am}\,u\Delta\,\mathrm{am}\left(\frac{2K'}{n},k'\right)\right\}\left\{1-\sin\mathrm{am}\,u\Delta\,\mathrm{am}\left(\frac{4K'}{n},k'\right)\right\}\cdots\left\{1-\sin\mathrm{am}\,u\Delta\,\mathrm{am}\left(\frac{(n-1)K'}{n},k'\right)\right\}}{\left\{1+\sin\mathrm{am}\,u\Delta\,\mathrm{am}\left(\frac{2K'}{n},k'\right)\right\}\left\{1+\sin\mathrm{am}\,u\Delta\,\mathrm{am}\left(\frac{4K'}{n},k'\right)\right\}\cdots\left\{1+\sin\mathrm{am}\,u\Delta\,\mathrm{am}\left(\frac{(n-1)K'}{n},k'\right)\right\}}$$

$$\sqrt{\frac{1-\lambda_1\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)}{1+\lambda_1\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)}}$$

$$= \sqrt{\frac{1-k\sin\mathrm{am}\,u}{1+k\sin\mathrm{am}\,u}}\cdot\frac{\left\{1-\Delta\,\mathrm{am}\left(\frac{K'}{n},k'\right)\sin\mathrm{am}\,u\right\}\left\{1-\Delta\,\mathrm{am}\left(\frac{3K'}{n},k'\right)\sin\mathrm{am}\,u\right\}\cdots\left\{1-\Delta\,\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)\sin\mathrm{am}\,u\right\}}{\left\{1+\Delta\,\mathrm{am}\left(\frac{K'}{n},k'\right)\sin\mathrm{am}\,u\right\}\left\{1+\Delta\,\mathrm{am}\left(\frac{3K'}{n},k'\right)\sin\mathrm{am}\,u\right\}\cdots\left\{1+\Delta\,\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)\sin\mathrm{am}\,u\right\}}$$

$$\frac{\lambda_1}{kM_1}\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \sin\mathrm{am}\,u + \frac{2}{k}\sum\frac{\Delta\,\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin\mathrm{am}\,u}{\sin^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\mathrm{am}\,u}$$

$$\frac{(-1)^{\frac{n-1}{2}}\lambda_1}{kM_1}\cos\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \cos\mathrm{am}\,u - \frac{2(-1)^{\frac{n-1}{2}}}{k}\sum\frac{(-1)^q\sin\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\Delta\,\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\cos\mathrm{am}\,u}{\sin^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\mathrm{am}\,u}$$

$$\frac{(-1)^{\frac{n-1}{2}}}{M_1}\Delta\,\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \Delta\,\mathrm{am}\,u \quad -2(-1)^{\frac{n-1}{2}}\sum\frac{(-1)^q\sin\mathrm{am}\,\frac{(2q-1)iK'}{n}\Delta\,\mathrm{am}\,u}{\sin^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\mathrm{am}\,u}$$

$$\frac{\lambda_1'}{k'M_1}\tan\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) = \tan\mathrm{am}\,u + 2\sum\frac{(-1)^q\cos\mathrm{am}\left(\frac{2qK'}{n},k'\right)\Delta\,\mathrm{am}\left(\frac{2qK'}{n},k'\right)\sin\mathrm{am}\,u\cos\mathrm{am}\,u}{1-\Delta^2\,\mathrm{am}\left(\frac{2qK'}{n},k'\right)\sin^2\mathrm{am}\,u}.$$

In the formulas for the first transformation it was put $(-1)^{\frac{n-1}{2}}M$ instead of m. It was convenient to exhibit the formulas for the second transformation in two ways, both under inmaginary and real form, in which additionally everywhere instead of $k\sin\mathrm{am}\,\frac{2miK'}{n}$, $k\cos\mathrm{am}\,\frac{2miK'}{n}$ etc. it was written $\frac{1}{\sin\mathrm{am}\,\frac{(n-2m)miK'}{n}}$, $\frac{1}{\cos\mathrm{am}\,\frac{(n-2m)miK'}{n}}$ etc.: this was, as the reduction to the real form, easily done by means of the formulas in §19. Where the ambiguous sign $\pm$ was put, the first $+$ is to be chosen, if $\frac{n-1}{2}$ is an even number, the other $-$, if $\frac{n-1}{2}$ is an odd number; the contrary holds for the sign $\mp$. In the sums denoted by the prefixed sign $\sum$ the values $1, 2, 3, \cdots, \frac{n-1}{2}$ are to be assigned to the number q.

From the formulas propounded for the first transformation it is clear that, if n successively becomes:

$$0, \quad \frac{K}{n}, \quad \frac{2K}{n}, \quad \frac{3K}{n}, \quad \frac{4K}{n}, \cdots,$$

$\mathrm{am}\left(\frac{u}{M},\lambda\right)$ will be:

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi, \cdots,$$

whence we obtain:

$$\frac{K}{nM} = \Lambda.$$

On the other hand, we have seen in the second transformation, if u becomes: $0,\ K,\ 2K,\ 3K, \cdots$ or $\mathrm{am}\,u$: $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, \cdots$, that also $\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)$ becomes: $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, \cdots$,, whence in this case:

$$\frac{K}{M_1} = \Lambda_1.$$

By the way, it is plain from the formulas for the moduli λ, λ', λ_1, λ'_1 while n grows that the moduli λ, λ'_1 rapidly converge to zero, and hence at the same time the moduli λ', λ_1 get very close to unity. Therefore, it is convenient to call the first transformation of the modulus *transformation of the larger to the smaller*, the second *transformation of the smaller into the greater*.

1.14 On complementary Transformations or how from the Transformation of one Modulus into another the Transformation of one Complement into another is derived

25.

In the formulas found above:

$$\tan\operatorname{am}\sqrt{\frac{k'^n}{\lambda'}}\tan\operatorname{am} u\tan\operatorname{am}(u+4\omega)\tan\operatorname{am}(u+8\omega)\cdots\tan\operatorname{am}(u+4(n-1)\omega)$$

let us put $u = iu'$, $\omega = i\omega'$, so that it is $\omega = \frac{mK+m'iK'}{n}$, $\omega' = \frac{m'K'-miK}{n}$. But it is (§ 19):

$$\begin{aligned}\tan\operatorname{am}(iu',k) &= i\sin\operatorname{am}(u',k')\\ \tan\operatorname{am}(iu',\lambda) &= i\sin\operatorname{am}(u',\lambda'),\end{aligned}$$

whence we see the quoted formula to go over into the following:

$$\sin\operatorname{am}\left(\frac{u'}{M},\lambda'\right) = (-1)^{\frac{n-1}{2}}\sin\operatorname{am} u'\sin\operatorname{am}(u'+4\omega')\sin\operatorname{am}(u'+8\omega')\cdots\sin\operatorname{am}(u'+4(n-1)\omega')\ (\text{mod. } k')$$

Further, we found:

$$\begin{aligned}\lambda' &= \frac{k'^n}{[\Delta\operatorname{am}2\omega\Delta\operatorname{am}4\omega\cdots\Delta\operatorname{am}4(n-1)\omega]^4}\\ M &= (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}(n-1)\omega]^2}\end{aligned}$$

which from the formulas:

$$\begin{aligned}\Delta\,\mathrm{am}(iu,k) &= \frac{1}{\sin\mathrm{coam}(u,k')}\\ \sin\mathrm{coam}(iu,k) &= \frac{1}{\Delta\,\mathrm{am}(u,k')},\end{aligned}$$

whence it also follows:

$$\frac{\sin\mathrm{coam}(iu,k)}{\sin\mathrm{am}(iu,k)} = \frac{-i}{\tan\mathrm{am}(u,k')\Delta\,\mathrm{am}(u,k')} = \frac{-i\sin\mathrm{coam}(u,k')}{\sin\mathrm{am}(u,k')}$$

go over into the following:

$$\begin{aligned}\lambda' &= k'^n[\sin\mathrm{coam}\,2\omega'\sin\mathrm{coam}\,4\omega'\sin\mathrm{coam}(n-1)\omega']^4 \quad (\text{mod. } k')\\ M &= \frac{[\sin\mathrm{coam}\,2\omega'\sin\mathrm{coam}\,4\omega'\sin\mathrm{coam}(n-1)\omega']^2}{[\sin\mathrm{am}\,2\omega'\sin\mathrm{am}\,4\omega'\sin\mathrm{am}(n-1)\omega']^2} \quad (\text{mod. } k')\end{aligned}$$

Having compared these formulas to those which serve for the transformation of the modulus k into the modulus λ:

$$\begin{aligned}\sin\mathrm{am}\left(\frac{u}{M},\lambda\right) &= \sqrt{\frac{k^n}{\lambda}}\sin\mathrm{am}\,u\sin\mathrm{am}(u+4\omega)\sin\mathrm{am}(u+8\omega)\cdots\sin\mathrm{am}(u+4(n-1)\omega)\\ \lambda &= k^n[\sin\mathrm{coam}\,2\omega\sin\mathrm{coam}\,4\omega\cdots\sin\mathrm{coam}(n-1)\omega]^4\\ M &= (-1)^{\frac{n-1}{2}}\frac{[\sin\mathrm{coam}\,2\omega\sin\mathrm{coam}\,4\omega\cdots\sin\mathrm{coam}(n-1)\omega]^2}{[\sin\mathrm{am}\,2\omega\sin\mathrm{am}\,4\omega\cdots\sin\mathrm{am}(n-1)\omega]^2}\end{aligned}$$

it reveals a theorem, which has to be considered to be of highest importance in the transformation theory:

Which formulas on the transformation of the modulus k into the modulus λ whatsoever can be propounded, the same hold, having changed k into k', λ into λ', ω into $\omega' = \frac{\omega}{i}$, M into $(-1)^{\frac{n-1}{2}}M$.

But we will call the transformation of the complement into another complement, derived from the propounded transformation in the way just described, *complementary transformation.*

It easily becomes clear that the real transformations of the modulus k' are the complementary one of the real transformation of the modulus k, such that nevertheless the second of the modulus k' is the complementary of the first of the modulus k, and the first of the modulus k' is the complementary of the second of the modulus k. For, if in the just propounded theorem one puts $\omega = \frac{\pm K}{n}$, $\omega = \frac{\pm iK'}{n}$ which corresponds to the first and second transformations of the modulus k, it is $\omega' = \frac{\omega}{i} = \frac{\mp iK}{n}$, $\omega' = \frac{\omega}{i} = \frac{\pm K'}{n}$, which corresponds to the second and first transformations of the modulus k', respectively. Because while the modulus grows the complement decreases and vice versa, if the transformation of the complement into the complement is the transformation of the greater into the smaller, the transformation of the complement or the complementary transformation must be the one of the greater into the smaller and vice versa. Therefore, we see, having changed k into k', that λ goes over into λ_1', λ_1 goes over into λ'. Only the multiplicator M, common to the first transformation and its complementary counterpart, will go over into M_1, which belongs to the second transformation and its complementary counterpart, and vice versa M_1 into M. Hence, from the formulas found above:

$$\Lambda = \frac{K}{nM}, \quad \Lambda_1 = \frac{K}{M_1}$$

these ones follows:

$$\Lambda_1' = \frac{K'}{nM_1'}, \quad \Lambda' = \frac{K'}{M},$$

whence these formulas of highest importance in this emerge:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}; \quad \frac{\Lambda_1'}{\Lambda_1} = \frac{1}{n} \cdot \frac{K'}{K}.$$

These formulas define the genuine character of the propounded transformation, whence it is clear that we justly referred the particular transformations to the particular numbers n. I mention, if n was a composite number $= n'n''$, that from the particular real roots of the modular equations or from the particular real moduli into which the given modulus k can be transformed by means of a substitution of n-th order, one reaches equations of this kind:

$$\frac{\Lambda'}{\Lambda} = \frac{n'}{n'} \cdot \frac{K'}{K},$$

which correspond to the particular factorizations of the number n into two factors. Therefore, from their total number, if n was a square, it will also be:

$$\frac{\Lambda'}{\Lambda} = \frac{K'}{K}, \quad \text{whence} \quad \lambda = k,$$

which tells us, in the case in which n is a square, that from the total number of substitutions there is one which yields a multiplication.

1.15 On Transformations supplementary for Multiplication

26.

Let us recall the formulas:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}, \quad \frac{\Lambda_1'}{\Lambda_1} = \frac{1}{n} \cdot \frac{K'}{K},$$

having written which in this way:

$$\begin{aligned} \frac{\Lambda'}{\Lambda} &= n\frac{K'}{K} \\ \frac{K'}{K} &= n\frac{\Lambda_1'}{\Lambda_1}, \end{aligned}$$

it becomes clear *that the modulus λ in the same way depends on the modulus k as the modulus k depends on the modulus λ_1, or the the modulus k depends in the same way on the modulus λ as the modulus λ_1 depends on the modulus k.* Therefore, by means of the first transformation or of the greater into the smaller, in which k is transformed into λ, λ_1 will be transformed into k; by means of the second transformation or the of the smaller into the greater, in which k is transformed into λ_1, λ will be transformed into k. Therefore, *after the first transformation after having used the second or after the second after having used the first the modulus k is transformed into itself, or the first and the second transformation applied successively, in an arbitrary order, yield a multiplication.*

Let us call M' the multiplicator which depends on λ in the same way as M_1 depends on a, and let us call M_1 the multiplicator which depends in the same way on λ_1 as M depends on k such that the following equations are obtained:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$
$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{dy}{M'\sqrt{(1-y^2)(1-k^2y^2)}},$$

of which the one corresponds to the transformation of the modulus k into the modulus λ by means of the first transformation, the other to the transformation of the modulus λ into the modulus k by means of the second transformation. From these equations it arises:

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{dx}{MM'\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{whence} \quad z = \sin\operatorname{am}\left(\frac{u}{MM'}\right).$$

But from equation $\Lambda_1 = \frac{K}{M_1}$ by changing k into λ having done which K goes over into Λ, λ_1 into k, Λ_1 into K, M_1 into M' one obtains $K = \frac{\Lambda}{M'}$ having compared which equation to $\Lambda = \frac{K}{nM}$ it arises $\frac{1}{MM'} = n$, whence:

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{ndx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

In the same from the equation $\Lambda = \frac{K}{nM}$ by changing k into λ_1 having done which K goes over into Λ_1, λ into k, Λ into K, M_1 into M_1', $K = \frac{\Lambda_1}{nM_1'}$, after having compared which equation to $\Lambda_1 = \frac{K}{M_1}$ this yields $\frac{1}{M_1M_1'} = n$; hence, we see that in those two cases after two successively applied transformations the argument is multiplied by the number n.

If after the transformation of the modulus k into the modulus λ λ is then again transformed back into the modulus k such that a multiplications arises we will call this transformation *the supplementary for multiplication* of the latter or simply *supplementary*.

Let us both for the sake of an example and the use for the following list the formulas for the *supplementary* transformation of the *first* or of the modulus λ into the modulus k which transformation will be the second of λ; we will nevertheless only list them in the imaginary form, since the reduction to the real one is easily done. We immediately obtain these formulas, if in those which were propounded above on the second transformation of the modulus k (confer Table II, A. §24) we put λ instead of k, k instead of λ_1, $\frac{u}{M}$ instead of u, $M = \frac{1}{nM}$ instead of M_1, whence $\frac{u}{MM'} = nu$ instead of $\frac{u}{M_1}$. In these formulas, but only in those, the modulus λ will be the right one, if the modulus k is not explicitly added; furthermore, for the sake of brevity it was put $y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)$; to the number q as above one has to assign the values:

$$1, 2, 3, \cdots, \frac{n-1}{2}.$$

1.16 Formulas for the Transformation of the Modulus λ into the Modulus k or the supplementary of the first

27.

$$k = \lambda^n \left\{ \sin\operatorname{coam}\frac{2i\Lambda'}{n} \sin\operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n} \right\}^4$$

$$k' = \frac{\lambda'^n}{\left\{ \Delta\operatorname{am}\frac{2i\Lambda'}{n} \Delta\operatorname{am}\frac{4i\Lambda'}{n} \cdots \Delta\operatorname{am}\frac{(n-1)i\Lambda'}{n} \right\}^4}$$

$$\frac{1}{nM}(-1)^{\frac{n-1}{2}} \left\{ \frac{\sin\operatorname{coam}\frac{2i\Lambda'}{n} \sin\operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}{\sin\operatorname{am}\frac{2i\Lambda'}{n} \sin\operatorname{am}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{am}\frac{(n-1)i\Lambda'}{n}} \right\}^2$$

$$\begin{aligned}
\sin\operatorname{am}(nu,k) &= \frac{nMy\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{2i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{4i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)} \\
&= \sqrt{\frac{\lambda^n}{k}} \sin\operatorname{am}\frac{u}{M} \sin\operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \sin\operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \sin\operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right) \\
\cos\operatorname{am}(nu,k) &= \frac{\sqrt{1-y^2}\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)} \\
&= \sqrt{\frac{k'\lambda^{n\,n}}{k\lambda'}} \cos\operatorname{am}\frac{u}{M} \cos\operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \cos\operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \cos\operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right) \\
\Delta\operatorname{am}(nu,k) &= \frac{\sqrt{1-\lambda^2y^2}\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}{\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)} \\
&= \sqrt{\frac{k'}{\lambda'^n}} \Delta\operatorname{am}\frac{u}{M} \cos\operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \Delta\operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \Delta\operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right)
\end{aligned}$$

$$\sqrt{\frac{1-\sin\operatorname{am}(nu,k)}{1+\sin\operatorname{am}(nu,k)}} = \sqrt{\frac{1-y}{1+y}} \cdot \frac{\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}$$

$$\sqrt{\frac{1-k\sin\operatorname{am}(nu,k)}{1+k\sin\operatorname{am}(nu,k)}} = \sqrt{\frac{1-\lambda y}{1+\lambda y}} \cdot \frac{\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}{\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}$$

$$\sin\operatorname{am}(nu,k) = \frac{\lambda y}{knM} - \frac{2y}{knM}\sum \frac{\cos\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\Delta\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n} - y^2}$$

$$\cos\operatorname{am}(nu,k) = \frac{(-1)^{\frac{n-1}{2}}\lambda\sqrt{1-y^2}}{knM} + \frac{2\sqrt{1-y^2}}{iknM}\sum \frac{(-1)^q \sin\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\Delta\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n} - y^2}$$

$$\Delta\operatorname{am}(nu,k) = \frac{(-1)^{\frac{n-1}{2}}}{nM}\sqrt{1-\lambda^2y^2} + \frac{2\sqrt{1-\lambda^2y^2}}{inM}\sum \frac{(-1)^q \sin\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\cos\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n} - y^2}$$

$$\tan\operatorname{am}(nu,k) = \frac{\lambda'}{k'nM}\cdot\frac{y}{\sqrt{1-y^2}} + \frac{2\lambda' y\sqrt{1-y^2}}{k'nM}\sum \frac{(-1)^q\Delta\operatorname{am}\frac{2qi\Lambda'}{n}}{\cos^2\operatorname{am}\frac{2qi\Lambda'}{n} - y^2\Delta^2\operatorname{am}\frac{2qi\Lambda'}{n}}.$$

This general analytical theorem concerning that supplementary transformation of the first I already communicated to Legendre at the beginning of August 1827, which he also wanted to mention in the note mentioned above (*Nova Astronomica* a. 1827, Nr. 130). A similar system of formulas for the other supplementary transformation of the second or the transformation of the modulus λ into the modulus could have been stated. To make all these thing more clear, it was convenient to give a complete list of the fundamental formulas for the first and second transformation and their complementary and supplementary one in the added table.

Only one of the total number of imaginary transformations has a supplementary one to multiplication. Let us suppose, which is possible, that the numbers m, m' §. 20 do not have a common factor: Further, let $m\mu' - \mu m' = 1$ μ, μ' denoting whole positive or negative integers. Now, if in our general formulas propounded on the transformation in § 20 one puts $\omega = \frac{\nu K + \mu' iK'}{nM}$ and k and λ are interchanged, one obtains formulas extending to the supplementary of the transformation. Having put $m = 1$, $m' = 0$ it is $\mu = 0$, $\mu' = 1$, whence $\frac{\mu K + \mu' iK'}{nM} = \frac{iK'}{nM} = \frac{i\Lambda'}{n}$ which yields the supplementary of the first, as saw.

A. First Transformation with Supplementary

$$(a)\qquad \lambda = k^n \sin^4\operatorname{coam}\frac{2K}{n}\sin^4\operatorname{coam}\frac{4K}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)K}{n}\qquad (\text{mod. } k)$$

$$(aa)\qquad k^n = \lambda^n \sin^4\operatorname{coam}\frac{2i\Lambda'}{n}\sin^4\operatorname{coam}\frac{4i\Lambda'}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)i\Lambda'}{n}\qquad (\text{mod. } \lambda)$$

$$= \frac{\lambda^n}{\Delta^4\operatorname{am}\frac{2\Lambda'}{n}\Delta^4\operatorname{am}\frac{4\Lambda'}{n}\cdots\Delta^4\operatorname{am}\frac{(n-1)\Lambda'}{n}}\qquad (\text{mod. } \lambda')$$

$$(b)\qquad M = \frac{\sin^2\operatorname{coam}\frac{2K}{n}\sin^2\operatorname{coam}\frac{4K}{n}\cdots\sin^2\operatorname{coam}\frac{(n-1)K}{n}}{\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}\frac{4K}{n}\cdots\sin^2\operatorname{am}\frac{(n-1)K}{n}}\qquad (\text{mod. } (k)$$

$$(bb)\qquad \frac{1}{nM} = \frac{\sin^2\operatorname{coam}\frac{2\Lambda'}{n}\sin^2\operatorname{coam}\frac{4\Lambda'}{n}\cdots\sin^2\operatorname{coam}\frac{(n-1)\Lambda'}{n}}{\sin^2\operatorname{am}\frac{2\Lambda'}{n}\sin^2\operatorname{am}\frac{4\Lambda'}{n}\cdots\sin^2\operatorname{am}\frac{(n-1)\Lambda'}{n}}\qquad (\text{mod. } \lambda')$$

$$\sin\operatorname{am}(u,k) = x;\quad \sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = y;\quad \sin\operatorname{am}(nu,k) = z$$

$$(c)\qquad y = (-1)^{\frac{n-1}{2}}\sqrt{\frac{k^n}{\lambda}}\sin\operatorname{am}u\sin\operatorname{am}\left(u+\frac{4K}{n}\right)\sin\operatorname{am}\left(u+\frac{8K}{n}\right)\sin\operatorname{am}\cdots\left(u+\frac{4(n-1)K}{n}\right)\qquad (\text{mod. } k)$$

$$= \frac{\frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{2K}{n}}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1-k^2x^2\sin^2\operatorname{am}\frac{2K}{n}\right)\left(1-k^2x^2\sin^2\operatorname{am}\frac{4K}{n}\right)\cdots\left(1-k^2x^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\right)}\qquad (\text{mod. } k)$$

$$(cc)\qquad z = \sqrt{\frac{\lambda^n}{k}}\sin\operatorname{am}\frac{u}{M}\sin\operatorname{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\sin\operatorname{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\cdots\sin\operatorname{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right)\qquad (\text{mod. } \lambda)$$

$$= \frac{nMy\left(1+\frac{y^2}{\tan^2\operatorname{am}\frac{2\Lambda'}{n}}\right)\left(1+\frac{y^2}{\tan^2\operatorname{am}\frac{4\Lambda'}{n}}\right)\cdots\left(1+\frac{y^2}{\tan^2\operatorname{am}\frac{(n-1)\Lambda'}{n}}\right)}{\left(1+\lambda^2y^2\tan^2\operatorname{am}\frac{2\Lambda'}{n}\right)\left(1+\lambda^2y^2\tan^2\operatorname{am}\frac{4\Lambda'}{n}\right)\cdots\left(1+\lambda^2y^2\tan^2\operatorname{am}\frac{(n-1)\Lambda'}{n}\right)}\qquad (\text{mod. } \lambda')$$

Complementary Transformations

$$(a)\qquad \lambda' = k'^n \sin^4\operatorname{coam}\frac{2iK}{n}\sin^4\operatorname{coam}\frac{4iK}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)iK}{n}\qquad (\text{mod. } k')$$

$$= \frac{k'^n}{\Delta^4\operatorname{am}\frac{2K}{n}\Delta^4\operatorname{am}\frac{4K}{n}\cdots\Delta^4\operatorname{am}\frac{(n-1)K}{n}}\qquad (\text{mod. } k)$$

$$(aa)\qquad k' = \lambda'^n \sin^4\operatorname{coam}\frac{2\Lambda'}{n}\sin^4\operatorname{coam}\frac{4\Lambda'}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)\Lambda'}{n}\qquad (\text{mod. } \lambda')$$

(b) and (bb) are the same as above.

$$\sin\operatorname{am}(u,k') = x;\quad \sin\operatorname{am}\left(\frac{u}{M},\lambda'\right) = y;\quad \sin\operatorname{am}(nu,k') = z$$

$$(c) \quad y = \sqrt{\frac{k'^n}{\lambda'}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4iK}{n}\right) \sin\operatorname{am}\left(u + \frac{8iK}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)iK}{n}\right) \quad (\text{mod. } k')$$

$$= \frac{\frac{x}{M}\left(1 + \frac{x^2}{\tan^2\operatorname{am}\frac{2K}{n}}\right)\left(1 + \frac{x^2}{\tan^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1 + \frac{x^2}{\tan^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1 + k'^2x^2\tan^2\operatorname{am}\frac{2K}{n}\right)\left(1 + k'^2x^2\tan^2\operatorname{am}\frac{4K}{n}\right)\cdots\left(1 + k'^2x^2\tan^2\operatorname{am}\frac{(n-1)K}{n}\right)} \quad (\text{mod. } k)$$

$$(cc) \quad z = (-1)^{\frac{n-1}{2}} \sin\operatorname{am}\frac{u}{M} \sin\operatorname{am}\left(\frac{u}{M} + \frac{4\Lambda'}{n}\right)\left(\frac{u}{M} + \frac{8\Lambda'}{n}\right)\cdots\left(\frac{u}{M} + \frac{4(n-1)\Lambda'}{n}\right) \quad (\text{mod. } \lambda')$$

$$= \frac{nMy\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{2\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{4\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-1)\Lambda'}{n}}\right)}{\left(1 - \lambda'^2y^2\sin^2\operatorname{am}\frac{2\Lambda'}{n}\right)\left(1 - \lambda'^2y^2\sin^2\operatorname{am}\frac{4\Lambda'}{n}\right)\cdots\left(1 - \lambda'^2y^2\sin^2\operatorname{am}\frac{(n-1)\Lambda'}{n}\right)} \quad (\text{mod. } \lambda')$$

$$\Lambda = \frac{K}{nM}; \quad \Lambda' = \frac{K'}{M}$$

B. Second Transformation with Supplementary

$$(a) \quad \lambda_1 = k^n \sin^4\operatorname{coam}\frac{2iK'}{n}\sin^4\operatorname{coam}\frac{4iK'}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)iK'}{n} \quad (\text{mod. } k)$$

$$= \frac{k^n}{\Delta^4\operatorname{am}\frac{2K'}{n}\Delta^4\operatorname{am}\frac{4K'}{n}\cdots\Delta^4\operatorname{am}\frac{(n-1)K'}{n}} \quad (\text{mod. } k')$$

$$(aa) \quad k = \lambda_1^n \sin^4\operatorname{coam}\frac{2\Lambda_1}{n}\sin^4\operatorname{coam}\frac{2\Lambda_1}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)\Lambda_1}{n} \quad (\text{mod. } \lambda_1)$$

$$(b) \quad M_1 = \frac{\sin^2\operatorname{coam}\frac{2K'}{n}\sin^2\operatorname{coam}\frac{4K'}{n}\cdots\sin^2\operatorname{coam}\frac{(n-1)K'}{n}}{\sin^2\operatorname{am}\frac{2K'}{n}\sin^2\operatorname{am}\frac{4K'}{n}\cdots\sin^2\operatorname{am}\frac{(n-1)K'}{n}} \quad (\text{mod. } k')$$

$$(bb) \quad \frac{1}{nM_1} = \frac{\sin^2\operatorname{coam}\frac{2\Lambda_1}{n}\sin^2\operatorname{coam}\frac{4\Lambda_1}{n}\cdots\sin^2\operatorname{coam}\frac{(n-1)\Lambda_1}{n}}{\sin^2\operatorname{am}\frac{2\Lambda_1}{n}\sin^2\operatorname{am}\frac{4\Lambda_1}{n}\cdots\sin^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}} \quad (\text{mod. } \lambda_1)$$

$$\sin\operatorname{am}(u,k) = x; \quad \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = y; \quad \sin\operatorname{am}(nu,k) = z$$

$$(c) \quad y = \sqrt{\frac{k^n}{\lambda_1}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4iK'}{n}\right)\sin\operatorname{am}\left(u + \frac{8iK'}{n}\right)\cdots\sin\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right) \quad (\text{mod. } k)$$

$$= \frac{\frac{x}{M_1}\left(1 + \frac{x^2}{\tan^2\operatorname{am}\frac{2K'}{n}}\right)\left(1 + \frac{x^2}{\tan^2\operatorname{am}\frac{4K'}{n}}\right)\cdots\left(1 + \frac{x^2}{\tan^2\operatorname{am}\frac{(n-1)K'}{n}}\right)}{\left(1 + k^2x^2\tan^2\operatorname{am}\frac{2K'}{n}\right)\left(1 + k^2x^2\tan^2\operatorname{am}\frac{4K'}{n}\right)\cdots\left(1 + k^2x^2\tan^2\operatorname{am}\frac{(n-1)K'}{n}\right)} \quad (\text{mod. } k')$$

$$(cc) \quad z = (-1)^{\frac{n-1}{2}}\sqrt{\frac{\lambda_1^n}{k}} \sin\operatorname{am}\frac{u}{M_1}\sin\operatorname{am}\left(\frac{u}{M_1} + \frac{4\Lambda_1}{n}\right)\sin\operatorname{am}\left(\frac{u}{M_1} + \frac{8\Lambda_1}{n}\right)\cdots\sin\operatorname{am}\left(\frac{u}{M_1} + \frac{4(n-1)\Lambda_1}{n}\right) (\text{mod. } \lambda_1)$$

$$= \frac{nM_1y\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{2\Lambda_1}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{4\Lambda_1}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}}\right)}{\left(1 - \lambda_1^2y^2\sin^2\operatorname{am}\frac{2\Lambda_1}{n}\right)\left(1 - \lambda_1^2y^2\sin^2\operatorname{am}\frac{4\Lambda_1}{n}\right)\cdots\left(1 - \lambda_1^2y^2\sin^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}\right)} \quad (\text{mod. } \lambda_1)$$

Complementary Transformations

$$(a) \quad \lambda_1' = k'^n \sin^4 \operatorname{coam} \frac{2K'}{n} \sin^4 \operatorname{coam} \frac{4K'}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)K'}{n} \qquad (\text{mod. } k')$$

$$(aa) \quad k' = \lambda_1'^n \sin^4 \operatorname{coam} \frac{2i\Lambda_1}{n} \sin^4 \operatorname{coam} \frac{4i\Lambda_1}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)i\Lambda_1}{n} \qquad (\text{mod. } \lambda_1')$$

$$= \frac{\lambda_1'^n}{\Delta^4 \operatorname{am} \frac{2\Lambda_1}{n} \Delta^4 \operatorname{am} \frac{2\Lambda_1}{n} \cdots \Delta^4 \operatorname{am} \frac{2\Lambda_1}{n}} \qquad (\text{mod. } \lambda_1)$$

(b) and (bb) are the same as above.

$$\sin\operatorname{am}(u, k') = x; \quad \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1'\right) = y; \quad \sin\operatorname{am}(nu, k') = z$$

$$(c) \quad y = (-1)^{\frac{n-1}{2}} \sqrt{\frac{k'^n}{\lambda_1'}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4K'}{n}\right) \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{8K'}{n}\right) \cdots \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4(n-1)K'}{n}\right) \quad (\text{mod. } k')$$

$$= \frac{\frac{x}{M_1}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{2K'}{n}}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4K'}{n}}\right) \cdots \left(1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{(n-1)K'}{n}}\right)}{\left(1 - k'^2 x^2 \sin^2 \operatorname{am} \frac{2K'}{n}\right)\left(1 - k'^2 x^2 \sin^2 \operatorname{am} \frac{4K'}{n}\right) \cdots \left(1 - k'^2 x^2 \sin^2 \operatorname{am} \frac{(n-1)K'}{n}\right)} \quad (\text{mod. } k')$$

$$(cc) \quad z = \sqrt{\frac{\lambda_1'^n}{k'}} \sin\operatorname{am} \frac{u}{M_1} \sin\operatorname{am}\left(\frac{u}{M_1} + \frac{4i\Lambda_1}{n}\right) \sin\operatorname{am}\left(\frac{u}{M_1} + \frac{8i\Lambda_1}{n}\right) \cdots \sin\operatorname{am}\left(\frac{u}{M_1} + \frac{4(n-1)i\Lambda_1}{n}\right) \quad (\text{mod. } \lambda_1')$$

$$= \frac{nM_1 y\left(1 + \frac{y^2}{\tan^2 \operatorname{am} \frac{2\Lambda_1}{n}}\right)\left(1 + \frac{y^2}{\tan^2 \operatorname{am} \frac{4\Lambda_1}{n}}\right) \cdots \left(1 + \frac{y^2}{\tan^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}}\right)}{\left(1 + \lambda_1'^2 y^2 \tan^2 \operatorname{am} \frac{2\Lambda_1}{n}\right)\left(1 + \lambda_1'^2 y^2 \tan^2 \operatorname{am} \frac{4\Lambda_1}{n}\right) \cdots \left(1 + \lambda_1'^2 y^2 \tan^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}\right)} \quad (\text{mod. } \lambda_1)$$

$$\Lambda_1 = \frac{K}{M_1}; \quad \Lambda_1' = \frac{K'}{nM_1}$$

1.17 General analytical Formulas for the Multiplication of elliptic Functions

28.

From two supplementary transformations it is possible to construct formulas for the multiplication or formulas by means of which the elliptic functions of the argument nu are expressed by elliptic functions of the argument u. To illustrate this with an example let us compose the multiplication from the first transformation and its supplementary one. For this purpose, recall the formula:

$$\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = (-1)^{\frac{n-1}{2}} \sqrt{\frac{k^n}{\lambda}} \sin\operatorname{am}\left(u + \frac{4K}{n}\right) \sin\operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)K}{n}\right)$$

which can also be represented in this way:

$$(-1)^{\frac{n-1}{2}} \sin \operatorname{am} \left(\frac{u}{M}, \lambda \right) = \sqrt{\frac{k^n}{\lambda}} \prod \sin \operatorname{am} \left(u + \frac{2mK}{n} \right)$$

while m denotes the numbers $0, \pm 1, \pm 2, \cdots, \pm \frac{n-1}{2}$. In this formula let us put $u + \frac{2m'iK'}{n}$ instead of u, whence $\frac{u}{M}$ goes over into $\frac{u}{M} + \frac{2m'iK'}{nM} = \frac{u}{M} + \frac{2m'i\Lambda'}{n}$: This yields:

$$(-1)^{\frac{n-1}{2}} \sin \operatorname{am} \left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda \right) = \sqrt{\frac{k^n}{\lambda}} \prod \sin \operatorname{am} \left(u + \frac{2mK + 2m'iK'}{n} \right).$$

Now, if also to m' the values $0, \pm 1, \pm 2, \cdots, \pm \frac{n-1}{2}$ are assigned, so that both m, m' take the values, after having taken the product we obtain:

$$(-1)^{\frac{n-1}{2}} \prod \sin \operatorname{am} \left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda \right) = \sqrt{\frac{k^{nn}}{\lambda}} \prod \sin \operatorname{am} \left(u + \frac{2mK + 2m'iK'}{n} \right),$$

where in the one product m', in the other both m and m' take all the values 0, $\pm 1, \pm 2, \cdots, \pm \frac{n-1}{2}$.

But we saw in the preceding § that it is:

$$\sin \operatorname{am}(nu, k) = \sqrt{\frac{\lambda^n}{k}} \sin \operatorname{am} \frac{u}{M} \sin \operatorname{am} \left(\frac{u}{M} + \frac{4i\Lambda'}{n} \right) \sin \operatorname{am} \left(\frac{u}{M} + \frac{8i\Lambda'}{n} \right) \cdots \sin \operatorname{am} \left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n} \right) \; (\text{mod. } \lambda),$$

which formula can also be represented this way:

$$(1.) \qquad \sin \operatorname{am}(nu, k) = \sqrt{\frac{\lambda^n}{k}} \prod \sin \operatorname{am} \left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda \right)$$

In the same way one finds:

$$(2.) \qquad \cos \operatorname{am} nu = \sqrt{\left(\frac{k}{k'} \right)^{nn-1}} \prod \cos \operatorname{am} \left(u + \frac{2mK + 2m'iK'}{n} \right)$$

$$(3.) \qquad \Delta \operatorname{am} nu = \sqrt{\left(\frac{1}{k'} \right)^{nn-1}} \prod \Delta \operatorname{am} \left(u + \frac{2mK + 2m'iK'}{n} \right).$$

These formulas are easily reduced to this form:

$$\text{(4.)}\qquad \sin\operatorname{am} nu = \sin\operatorname{am} u \prod \frac{1-\frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}}}{1-k^2\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}$$

$$\text{(5.)}\qquad \cos\operatorname{am} nu = \sin\operatorname{am} u \prod \frac{1-\frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{2mK+2m'iK'}{n}}}{1-k^2\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}$$

$$\text{(6.)}\qquad \Delta\operatorname{am} nu = n\Delta\operatorname{am} u \prod \frac{1-k^2\sin^2\operatorname{coam}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}{1-k^2\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}$$

It is convenient to add the following:

$$\text{(7.)}\qquad \prod \sin^2\operatorname{am}\frac{2mK+2m'iK'}{n} = \frac{(-1)^{\frac{n-1}{2}}n}{k^{\frac{nn-1}{2}}}$$

$$\text{(8.)}\qquad \prod \cos^2\operatorname{am}\frac{2mK+2m'iK'}{n} = \left(\frac{k'}{k}\right)^{\frac{nn-1}{2}}$$

$$\text{(9.)}\qquad \prod \Delta\operatorname{am}\frac{2mK+2m'iK'}{n} = k'^{\frac{nn-1}{2}}$$

In the six last formulas the number m only takes the positive values 0, 1, 2, 3, $\cdots$, $\frac{n-1}{2}$, nevertheless in such a way that if $m=0$ also to m' only the positive values 0, 1, 2, 3, $\cdots$, $\frac{n-1}{2}$ are assigned. These and other formulas for the multiplications were already propounded at first by Abel in a different but equivalent form, whence we were able to abbreviate on this subject.

1.18 On the Properties of modular Equations

29.

Since λ depends on k in the same way as k on λ_1 and λ_1' on k' in the same way as k' on λ': It is clear, if one constructs sequences of moduli which can be transformed into each other according to the same law where the one sequence contains the modulus k, the other its complement k' that in them the terms will occur in the same order one after another:

$$\begin{array}{c} \cdots, \lambda, \ k, \ \lambda_1, \ \cdots \\ \cdots, \lambda_1', k', \lambda', \ \cdots, \end{array}$$

this was already observed and proved by direct calculation in the transformations of second and third order first by Legendre. Since similar things hold for all transformed and imaginary moduli it is clear while λ denotes any arbitrary transformed modulus that the algebraic equations formed between k and λ or any between $u = \sqrt[4]{k}$ and $u = \sqrt[4]{v}$ which we called *modular equations* stay unchanged,

1.) if k and λ are interchanged
2.) if k' is put instead of k, λ' instead of λ.

This we already observed in the modular equations which belong to the transformations of third and fifth order:

$$\begin{aligned} &(1.) \qquad u^4 - v^4 + 2uv(1 - u^2v^2) = 0 \\ &(2.) \qquad u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0, \end{aligned}$$

and by means of clever observations exhibited algebraic formulas for the supplementary transformations. To also probe the other by means of examples, let us transform those equations into others between $kk = u^8$ and $\lambda\lambda = v^8$ which is not achieved without long calculation. Having performed them one obtains the equations:

$$\begin{aligned} &(1.) \qquad (k^2 - \lambda^2)^4 = 128k^2\lambda^2(1 - k^2)(1 - \lambda^2)(2 - k^2 - \lambda^2 + 2k^2\lambda^2) \\ &(2.) \qquad (k^2 - \lambda^2)^4 = 512k^2\lambda^2(1 - k^2)(1 - \lambda^2)(L - L'k^2 + L''k^4 - L'''k^6), \end{aligned}$$

if in the second it is put:

$$\begin{aligned} L &= 128 - 192\lambda^2 + 78\lambda^4 - 7\lambda^6 \\ L' &= 192 + 252\lambda^2 - 423\lambda^4 - 78\lambda^6 \\ L'' &= 78 + 423\lambda^2 - 252\lambda^4 - 192\lambda^6 \\ L''' &= 7 - 78\lambda^2 + 192\lambda^4 - 128\lambda^6 \end{aligned}$$

These equations go over into a much more convenient form having introduced $q = 1 - 2k^2$, $l = 1 - 2\lambda^2$. Having done the propounded equations become:

$$
\begin{aligned}
&(1.) \quad (q-l)^4 = 64\ (1-q^2)(1-l^2)[3+ql] \\
&(2.) \quad (q-l)^6 = 256(1-q^2)(1-l^2)[16ql(9-ql)^2 + 9(45-ql)(q-l)^2] \\
&\qquad\qquad\quad = 256(1-q^2)(1-l^2)[405(q^2+l^2) + 486ql - 9ql(q^2+l^2) - 270q^2l^2 + 16q^3l^3].
\end{aligned}
$$

These equations, if k' is put instead of k, λ' instead of λ, whence q goes over into $-q$, l into $-l$, remain unchanged, which was to be proved.

Corollary. Since we saw the propounded modular equations between $q = 1 - 2k^2$ and $l = 1 - 2\lambda^2$ to take a sufficiently convenient form it can also be interesting to expand the functions H and K' into a power series of the quantity q. This happens beautifully by the series:

$$
\begin{aligned}
K\ &= J\left(1 + \frac{q^2}{2\cdot 4} + \frac{5\cdot 5\cdot q^4}{2\cdot 4\cdot 6\cdot 8} + \frac{5\cdot 5\cdot 9\cdot 9\cdot q^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} + \cdots\right) \\
&\quad - \frac{\pi}{2J}\left(\frac{q}{2} + \frac{3\cdot 3\cdot q^3}{2\cdot 4\cdot 6} + \frac{3\cdot 3\cdot 7\cdot 7q^5}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \frac{3\cdot 3\cdot 7\cdot 7\cdot 11\cdot 11\cdot q^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} + \cdots\right) \\
K' &= J\left(1 + \frac{q^2}{2\cdot 4} + \frac{5\cdot 5\cdot q^4}{2\cdot 4\cdot 6\cdot 8} + \frac{5\cdot 5\cdot 9\cdot 9\cdot q^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} + \cdots\right) \\
&\quad + \frac{\pi}{2J}\left(\frac{q}{2} + \frac{3\cdot 3\cdot q^3}{2\cdot 4\cdot 6} + \frac{3\cdot 3\cdot 7\cdot 7q^5}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \frac{3\cdot 3\cdot 7\cdot 7\cdot 11\cdot 11\cdot q^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} + \cdots\right)
\end{aligned}
$$

where for the sake of brevity it was put:

$$\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}\sin^2\varphi}} = J.$$

As an easier task the equation for the transformation of third order:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$

can be transformed in such a way that the correlation between the modulus and the complements becomes clear. For, we obtain from that equation:

$$(1-u^4)(1+v^4) = 1-u^4v^4+2uv(1-u^2v^2) = (1-u^2v^2)(1+uv)^2$$
$$(1+u^4)(1-v^4) = 1-u^4v^4-2uv(1-u^2v^2) = (1-u^2v^2)(1-uv)^2,$$

having multiplied which equations it arises:

$$(1-u^8)(1-v^8) = (1-u^2v^2)^4.$$

Now let:

$$\begin{aligned} 1-u^8 &= k'k' \ = u'^8 \\ 1-v^8 &= \lambda'\lambda' = v'^8; \end{aligned}$$

having extracted the square root it is:

$$u'^2v'^2 = 1-u^2v^2,$$

or:

$$u^2v^2+u'^2v'^2 = \sqrt{k\lambda}+\sqrt{k'\lambda'} = 1,$$

which most elegant formulas already Legendre exhibited. And the formula is proved very elegantly by means of our analytical formulas, from which it follows in the case $n=3$:

$$\lambda = k^3 \sin^4 \operatorname{coam} 4\omega, \quad \lambda' = \frac{k'^3}{\Delta^4 \operatorname{am} 4\omega},$$

whence:

$$\begin{aligned} \sqrt{k\lambda} \ &= k^2 \sin^2 \operatorname{coam} 4\omega = \frac{k^2 \cos \operatorname{am} 4\omega}{\Delta^2 \operatorname{am} 4\omega} \\ \sqrt{k'\lambda'} &= \frac{k'^2}{\Delta \operatorname{am} 4\omega}, \end{aligned}$$

whence because it is:

$$k'k' + kk\cos^2 \operatorname{am} 4\omega = 1 - kk\sin^2 \operatorname{am} 4\omega = \Delta^2 \operatorname{am} 4\omega$$

we obtain, which was to be demonstrated:

$$\sqrt{k\lambda} + \sqrt{k'\lambda'} = 1.$$

To find a simpler equation between u, v, u', v' in the second example, I proceed as follows. I exhibit the propounded equation:

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

as follows:

$$(u^2 - v^2)(u^4 + 6u^2v^2 + v^4) + 4uv(1 - u^4v^4) = 0,$$

which is easily seen to take the following two forms:

$$\begin{aligned}(u^2 - v^2)(u+v)^4 &= -4uv(1 - u^4)(1 + v^4)\\ (u^2 - v^2)(u-v)^4 &= -4uv(1 + u^4)(1 - v^4),\end{aligned}$$

having multiplied which equations it arises:

$$(u^2 - v^2)^6 = 16u^2v^2(1 - u^8)(1 - v^8) = 16u^2v^2u'^8v'^8.$$

Because at the same time, as it was proved above, u^8 and v^8 go over into u'^8 and v'^8, respectively, we also obtain:

$$(v'^2 - u'^2)^6 = 16u'^2v'^2(1 - u'^8)(1 - v'^8) = 16u'^2v'^2u^8v^8.$$

Hence, having done the division and extracted the roots, it is found:

$$\frac{u^2 - v^2}{v'^2 - u'^2} = \frac{u'v'}{uv} \quad \text{or} \quad uv(u^2 - v^2) = u'v'(v'^2 - u'^2)$$

or:

$$\sqrt[4]{k\lambda} = (\sqrt{k} - \sqrt{\lambda}) = \sqrt[4]{k'\lambda'}(\sqrt{\lambda'} - \sqrt{k'}).$$

31.

There is even another extraordinary property of modular equations:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$
$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

which is found on first sight, of course that they remain unchanged, if instead of u, v one puts $\frac{1}{u}$, $\frac{1}{v}$. To show this in general for modular equations, let us note the following things, which can also be of use for other questions.

If one puts $y = kx$, one obtains:

$$\frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = \frac{kdx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

whence, since at the same time $x = 0$ as $y = 0$:

$$\int_0^y \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = k\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Hence, having put

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = u,$$

it is:

$$\int_0^y \frac{dy}{\sqrt{(1-y^2)(1-\frac{y^2}{k^2})}} = ku,$$

whence $x = \sin\operatorname{am}(u, k)$, $y = \sin\operatorname{am}\left(ku, \frac{1}{k}\right)$. Hence, the following equation arises:

$$\sin\operatorname{am}\left(ku, \frac{1}{k}\right) = k\sin\operatorname{am}(u, k),$$

whence also:

$$\begin{aligned}
\cos\mathrm{am}\left(ku, \frac{1}{k}\right) &= \Delta\,\mathrm{am}(u,k) \\
\Delta\,\mathrm{am}\left(ku, \frac{1}{k}\right) &= \cos\mathrm{am}(u,k) \\
\tan\mathrm{am}\left(ku, \frac{1}{k}\right) &= \frac{k}{k'}\cos\mathrm{coam}(u,k) \\
\sin\mathrm{coam}\left(ku, \frac{1}{k}\right) &= \frac{1}{\sin\mathrm{coam}(u,k)} \\
\cos\mathrm{coam}\left(ku, \frac{1}{k}\right) &= ik'\tan\mathrm{am}(u,k) \\
\Delta\,\mathrm{coam}\left(ku, \frac{1}{k}\right) &= \frac{ik'}{\cos\mathrm{am}(u,k)} \\
\tan\mathrm{coam}\left(ku, \frac{1}{k}\right) &= \frac{-i}{\cos\mathrm{coam}(u,k)}
\end{aligned}$$

Further, by putting iu instead of u, because the complement of the modulus $\frac{1}{k}$ becomes $\frac{ik'}{k}$, we obtain by means of the formulas in §19:

$$
\begin{aligned}
\sin\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \cos\operatorname{coam}(u, k') \\
\cos\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \sin\operatorname{coam}(u, k') \\
\Delta\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \frac{1}{\Delta\operatorname{coam}(u, k')} \\
\tan\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \cot\operatorname{coam}(u, k') \\
\sin\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \cos\operatorname{am}(u, k') \\
\cos\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \sin\operatorname{am}(u, k') \\
\sin\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \cos\operatorname{coam}(u, k') \\
\Delta\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \frac{\Delta\operatorname{am}(u, k')}{k} \\
\tan\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \cot\operatorname{am}(u, k').
\end{aligned}
$$

Now, let us investigate, what becomes of K, K' or $\arg.\operatorname{am}\left(\frac{\pi}{2}, k\right)$, $\arg.\operatorname{am}\left(\frac{\pi}{2}, k'\right)$, if one instead of k puts $\frac{1}{k}$; or let us investigate the values of the expressions $\arg.\operatorname{am}\left(\frac{\pi}{2}, k\right)$, $\arg.\operatorname{am}\left(\frac{\pi}{2}, k'\right)$, which would be in the notation used by Legendre: $F^1\left(\frac{1}{k}\right)$, $F^1\left(\frac{ik'}{k}\right)$. But, at first it is:

$$
\arg.\operatorname{am}\left(\frac{\pi}{2}, \frac{1}{k}\right) = \int\limits_0^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = \int\limits_0^k \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} + \int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}}.
$$

Having put $y = kx$ it is:

$$
\int\limits_0^k \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = k\int\limits_0^1 \frac{dx}{\sqrt{(1-y^2)(1-k^2x^2)}} = kK.
$$

To find the other integral $\int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}}$ let us put $y = \sqrt{1-k'k'x^2}$, whence $\frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = \frac{-kdx}{\sqrt{(1-x^2)(1-k'k'x^2)}}$. Now, because x therefore grows from 0 to 1, at the same time as y decreases from 1 to k, we obtain:

$$\int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = -i\int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(\frac{y^2}{k^2}-1\right)}} = i\int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)\left(1-k'k'x^2\right)}} = -ikK'$$

Hence, it arises:

$$\arg.\operatorname{am}\left(\frac{\pi}{2},\frac{1}{k}\right) = \left\{\arg.\operatorname{am}\left(\frac{\pi}{2},k\right) - i\arg.\operatorname{am}\left(\frac{\pi}{2},k'\right)\right\} = k\left\{K - iK'\right\},$$

or if k is changed into $\frac{1}{k}$, K goes over into $\{K - iK'\}$.

Secondly, having put $y = \cos\varphi$ it is:

$$\int\limits_0^1 \frac{dy}{\sqrt{(1-y^2)\left(1+\frac{k'k'}{kk}y^2\right)}} = k\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k'k'\sin^2\varphi}} = kK',$$

whence:

$$\arg.\operatorname{am}\left(\frac{\pi}{2},\frac{ik'}{k}\right) = k\arg.\operatorname{am}\left(\frac{\pi}{2},k'\right) = kK',$$

or if k is changed into $\frac{1}{k}$, K' goes over into kK'.

Therefore, in general, having changed k into $\frac{1}{k}$, $mK + im'K'$ goes over into $k\left\{mK + (m'-m)iK'\right\}$, whence $\sin\operatorname{coam}\left\{\frac{kp(mK+(m'-m)iK'}{n},k\right\}$ goes over into $\sin\operatorname{coam}\left\{\frac{p(mK+m'iK'}{n},k\right\}$ which from the formula

$$\sin\operatorname{coam}\left(ku,\frac{1}{k}\right) = \frac{1}{\sin\operatorname{coam}(u,k)}$$

becomes:

$$\sin\operatorname{coam}\left\{\frac{kp(mK+(m'-m)iK'}{n},\frac{1}{k}\right\}=\frac{1}{\sin\operatorname{coam}\left\{\frac{kp(mK+(m'-m)iK'}{n},k\right\}}.$$

Now, therefore having put $\omega=\frac{mK+m'iK'}{n}=\omega$, $\frac{mK+(m'-m)iK'}{n}=\omega_1$, the expression:

$$\lambda=k^n[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\sin\operatorname{coam}6\omega\cdots\sin\operatorname{coam}(n-1)\omega]^4,$$

having changed k into $\frac{1}{k}$ goes over into this one:

$$\frac{1}{k^n[\sin\operatorname{coam}2\omega_1\sin\operatorname{coam}4\omega_1\sin\operatorname{coam}6\omega_1\cdots\sin\operatorname{coam}(n-1)\omega_1]^4}=\frac{1}{\mu},$$

where μ itself is a root of a modular equation, or one of the number of moduli into which the propounded modulus k can be transformed by means of a transformation of nth order. For, from the values which ω can take that a transformed modulus arises, one will also be ω_1. Hence, also the reason is clear, why in general modular equations, having changed k into $\frac{1}{k}$, λ into $\frac{1}{\lambda}$ must remain unchanged.

Additionally, I mention, if according to the same law of transformation k is transformed into $k^{(m)}$, λ into $\lambda^{(m)}$ and if then $k^{(m)}$ is put instead of k, that also λ goes over into $\lambda^{(m)}$; hence, modular equations, if at the same time k is changed into $k^{(m)}$ as λ is changed into $\lambda^{(m)}$, have to they unchanged. So, for the sake of an example, the equation $\sqrt{k\lambda}+\sqrt{k'\lambda'}=1$ which is a modular equation for the transformation of third order, must remain unchanged, if instead of k, λ it is put $\frac{1-k'}{1+k'}$, $\frac{1-\lambda'}{1+\lambda'}$, respectively, whence one has to put $\frac{2\sqrt{k'}}{1+k'}$, $\frac{2\sqrt{\lambda'}}{1+\lambda'}$, which is known to be achieved by means of a transformation of second order. The equation $\sqrt{k\lambda}+\sqrt{k'\lambda'}=1$ goes over into this one:

$$\sqrt{\frac{(1-k')(1-\lambda')}{(1+k')(1+\lambda')}}+\frac{2\sqrt[4]{k'\lambda'}}{\sqrt{(1+k')(1+\lambda')}}=1,$$

or:

$$2\sqrt[4]{k'\lambda'}=\sqrt{(1+k')(1+\lambda')}-\sqrt{(1-k')(1-\lambda')}.$$

Having squared both sides it arises:

$$4\sqrt{k'\lambda'} = 2(1+k'\lambda') - 2k\lambda, \quad \text{or} \quad k\lambda = 1 + k'\lambda' - 2\sqrt{k'\lambda'},$$

which having extracted the roots reduces to the propounded one:

$$\sqrt{k\lambda} = 1 - \sqrt{k'\lambda'} \quad \text{or} \quad \sqrt{k\lambda} + \sqrt{k'\lambda'} = 1.$$

This example was already propounded by Legendre. But, it can be shown in general on t the composition of transformations that having used two or several transformations successively that one reaches the same, now matter in which order they are applied.

32.

But, among the properties of modular equations there is one which I consider to be most remarkable and outstanding, namely, *that they all satisfy the same differential equation of third order*. For its investigation it will nevertheless have to be expanded a bit.

It is well known having put:

$$aK + bK' = Q$$

that it will be:

$$k(1-k^2)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} = kQ,$$

a, b denoting arbitrary constants. Therefore, having also put:

$$a'K + b'K' = Q',$$

a', b' denoting other arbitrary constants, it will be:

$$k(1-k^2)\frac{d^2Q'}{dk^2} + (1-3k^2)\frac{dQ'}{dk} = kQ'.$$

Having combined these equations one obtains:

$$k(1-k^2)\left\{Q\frac{d^2Q'}{dk^2} - Q'\frac{d^2Q}{dk^2}\right\} + (1-3k^2)\left\{Q\frac{dQ'}{dk} - Q'\frac{dQ}{dk}\right\} = 0$$

whence after an integration:

$$k(1-k^2)\left\{Q\frac{dQ'}{dk}-Q'\frac{dQ}{dk}\right\}=(ab'-a'b)k(1-k^2)\left\{K\frac{dK'}{dk}-K'\frac{dK}{dk}\right\}=(ab'-a'b)C.$$

The constant C was already found from a special by Legendre to be $=-\frac{\pi}{2}$, whence now it is:

$$Q\frac{dQ'}{dk}-Q'\frac{dQ}{dk}=-\frac{1}{2}\frac{\pi(ab'-a'b)dk}{k(1-k^2)},$$

or

$$d\frac{Q'}{Q}=-\frac{1}{2}\frac{\pi(ab'-a'b)}{k(1-k^2)QQ}$$

Similarly, λ denoting another arbitrary modulus, having put

$$\alpha\Lambda+\beta\Lambda'=L,\qquad \alpha'\Lambda+\beta'\Lambda'=L',$$

it will be

$$d\frac{L}{L'}=-\frac{1}{2}\frac{\pi(\alpha\beta'-\alpha'\beta)d\lambda}{\lambda(1-\lambda\lambda)LL}.$$

Let λ be the modulus into which k is transformed by a first transformation of n-th order; further, let $Q=K$, $Q'=K'$, $K=\Lambda$, $L'=\Lambda'$; it will be:

$$\frac{L'}{L}=\frac{\Lambda'}{\Lambda}=\frac{nK'}{K}=\frac{nQ'}{Q},$$

whence:

$$\frac{ndk}{k(1-k^2)KK}=\frac{d\lambda}{\lambda(1-\lambda\lambda)\Lambda\Lambda}.$$

But we found for the transformation $\Lambda=\frac{K}{nM}$, whence:

$$MM=\frac{1}{n}\cdot\frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

In the second transformation, we saw that $\frac{\Lambda_1'}{\Lambda_1}=\frac{1}{n}\cdots\frac{K'}{K}$, $\Lambda_1=\frac{K}{M_1}$, whence:

$$\frac{dk}{k(1-k^2)KK}=\frac{nd\lambda_1}{\lambda_1(1-\lambda_1^2)\Lambda_1\Lambda_1},$$

whence also here:

$$M_1 M_1 = \frac{1}{n} \cdot \frac{\lambda_1(1-\lambda_1^2)dk}{k(1-k^2)d\lambda_1}.$$

But generally, no matter what the modulus λ is, whether real or imaginary, into which by means of a transformation of n-th order the propounded modulus k can be transformed, the following equation will hold:

$$MM = \frac{1}{n} \cdot \frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

To show this, I note that in general one obtains equations of the form:

$$\begin{aligned} \alpha\Lambda \quad + i\beta\Lambda' &= \frac{aK + ibK'}{nM} \\ \alpha'\Lambda' + i\beta'\Lambda &= \frac{a'K' + ib'K}{nM}, \end{aligned}$$

where a, a', α, α' denote odd numbers, b, b', β, β' denote even numbers, both either positive or negative of such a kind that $aa' + bb' = 1$, $\alpha\alpha' + \beta\beta' = 1$. Hence, having put:

$$\begin{aligned} aK + ibK' &= Q, \qquad & a'K' + ib'K &= Q' \\ a\Lambda + ib\Lambda' &= L, \qquad & \alpha'\Lambda' + i\beta'K &= L' \end{aligned}$$

we obtain, because $aa' + bb' = 1$, $\alpha\alpha' + \beta\beta' = 1$:

$$d\frac{Q'}{Q} = -\frac{1}{2}\frac{n\pi dk}{k(1-k^2)QQ'}, \qquad d\frac{L'}{L} = -\frac{1}{2}\frac{\pi d\lambda}{\lambda(1-\lambda^2)LL'},$$

whence, because it is:

$$\frac{Q'}{Q} = \frac{L'}{L} \quad L = \frac{Q}{nM},$$

it is in general:

$$MM = \frac{1}{n} \cdot \frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

I mention that the found equation can also be exhibited this way:

$$MM = \frac{1}{n} \cdot \frac{\lambda^2(1-\lambda^2)d(k^2)}{k^2(1-k^2))d(\lambda^2)} = \frac{1}{n} \cdot \frac{\lambda'^2(1-\lambda'^2)d(k'^2)}{k'^2(1-k'^2)d(\lambda'^2)},$$

whence we see that the expression MM is not changed, if instead of k, λ one puts k', λ', or what we demonstrated above that in complementary transformations, not taking into account the sign, the multiplicator M is the same. Further, by changing k into λ, λ into k, having done which the transformation goes over into the supplementary, MM is changed into

$$\frac{1}{n} \cdot \frac{k(1-k^2)d\lambda}{\lambda(1-\lambda^2)dk} = \frac{1}{nnMM}, \quad \text{or} \quad M \quad \text{into} \quad \frac{1}{nM},$$

what was already proved above.

33.

Having put $Q = aK + ibK'$, $L = \alpha\Lambda + i\beta\Lambda'$ it always possible to determine constants a, b, α, β such that $L = \frac{Q}{M}$ or $Q = ML$. Further, one has the equations:

$$\begin{aligned} &(1.) \qquad (k-k^3)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} - kQ = 0 \\ &(2.) \qquad (\lambda-\lambda^3)\frac{d^2Q}{d\lambda^2} + (1-3\lambda^2)\frac{dQ}{d\lambda} - \lambda Q = 0, \end{aligned}$$

which can also be represented this way:

$$\begin{aligned} &(3.) \qquad \frac{d}{dk}\left\{\frac{(k-k^3)dQ}{dk}\right\} - kQ = 0, \\ &(4.) \qquad \frac{d}{d\lambda}\left\{\frac{(\lambda-\lambda^3)dL}{d\lambda}\right\} - \lambda L = 0. \end{aligned}$$

Let us substitute in the equation:

$$(k-k^3)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} - kQ = 0,$$

$Q = ML$, it arises:

$$L\left\{(k-k^3)\frac{d^2M}{dk^2}+(1-3k^2)\frac{dM}{dk}-kM\right\}$$
$$+\frac{dL}{dk}\left\{2(k-k^3)\frac{dM}{dk}+(1-3k^2)M\right\}+(k-k^3)M\frac{d^2L}{d^2k}=0,$$

having multiplied which by M, we obtain:

$$(5.)\qquad LM\left\{(k-k^3)\frac{d^2M}{dk^2}+(1-3k^2)\frac{dM}{dk}-kM\right\}+\frac{d}{dk}\left\{\frac{(k-k^3)MMdL}{dk}\right\}=0.$$

But from the preceding § it is:

$$MM=\frac{(\lambda-\lambda^3)dk}{n(k-k^3)d\lambda},\quad\text{whence}\quad\frac{(k-k^3)MMdL}{dk}=\frac{(\lambda-\lambda^3)dL}{nd\lambda}.$$

Further, from equation (4.) it becomes:

$$d\left\{\frac{(\lambda-\lambda^3)}{d\lambda}\right\}=\lambda Ld\lambda,$$

whence:

$$\frac{d}{dk}\left\{\frac{(k-k^3)MMdL}{dk}\right\}=\frac{1}{n}\frac{d}{dk}\left\{\frac{(\lambda-\lambda^3)}{d\lambda}\right\}=\frac{\lambda Ld\lambda}{ndk}.$$

Hence, equation (5.) divided by L goes over into this one:

$$(6.)\qquad M\left\{(k-k^3)\frac{d^2M}{dk^2}+(1-3k^2)\frac{dM}{dk}-kM\right\}=\frac{\lambda d\lambda}{ndk}=0.$$

If in this equation the value of M from the equation $MM=\frac{(\lambda-\lambda^3)dk}{n(k-k^3)d\lambda}$ is substituted, one obtains a differential equation for the moduli k and λ themselves, which easily becomes clear to rise up to third order. Having performed the a little cumbersome calculation it is found:

$$(7.)\qquad \frac{3d^2\lambda^2}{dk^4}-\frac{2d\lambda}{dk}\cdot\frac{d^3\lambda}{dk^3}+\frac{d\lambda^2}{dk^2}\left\{\left[\frac{1+k^2}{k-k^3}\right]^2+\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk^2}\right\}=0.$$

In this equation dk is considered as a constant differential. If it pleases to transform it into another in which no differential is put constant one will have to put:

$$\begin{aligned}
\frac{d^2\lambda}{dk^2} &= \frac{d^2\lambda}{dk^2} - \frac{d\lambda d^2k}{dk^3} \\
\frac{d^3\lambda}{dk^3} &= \frac{d^3\lambda}{dk^3} - \frac{3d^2\lambda d^2k}{dk^4} - \frac{d\lambda d^3k}{dk^4} + \frac{3d\lambda d^2k^2}{dk^5}
\end{aligned}$$

whence:

$$\frac{3d^2\lambda^2}{dk^4} - \frac{2d\lambda}{dk}\frac{d^3\lambda}{dk^3} = \frac{3d^2\lambda^2}{dk^4} - \frac{3d\lambda^2 d^2k^2}{dk^6} + \frac{2d\lambda^2 d^3k}{dk^5} - \frac{2d\lambda d^3\lambda}{dk^4}.$$

Hence, equation (7.) multiplied by dk^6 goes over into the following in which no differential is put constant, or in which any arbitrary can be considered as such:

$$(8.)\quad 3\left\{dk^2d^2\lambda^2 - d\lambda^2d^2k^2\right\} - 2dkd\lambda\left\{dkd^3\lambda - d\lambda d^3k\right\} + dk^2d\lambda^2\left\{\left[\frac{1+k^2}{k-k^3}\right]^2 dk^2 - \left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2 d\lambda^2\right\} = 0.$$

This equation is clear, having interchanged the elements λ and k, to remain unchanged, what we proved above on modular equation.

It is worth the effort to investigate our differential equation of third order by another method. For this aim, let us into the equation from which we start:

$$(k-k^3)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} - kQ = 0$$

introduce the quantity:

$$(k-k^3)QQ = s.$$

It is

$$\begin{aligned}
\frac{ds}{dk} &= (1-3k^2)QQ + 2(k-k^3)Q\frac{dQ}{dk} \\
\frac{d^2s}{dk^2} &= -6kQQ + 4(1-3k^2)Q\frac{dQ}{dk} + 2(k-k^3)\left[\frac{dQ}{dk}\right]^2 + 2(k-k^3)Q\frac{d^2}{dk^2}.
\end{aligned}$$

If in the equation one puts:

$$(k-k^3)\frac{d^2Q}{dk^2} = kQ - (1-3k^2)\frac{dQ}{dk},$$

it arises:

$$\begin{aligned}\frac{d^2s}{dk^2} &= -4kQQ + 2(1-3k^2)Q\frac{dQ}{dk} + 2(k-k^3)\left[\frac{dQ}{dk}\right]^2 \\ &= 2\frac{dQ}{dk}\left\{(1-3k^2)Q + (k-k^3)\frac{dQ}{dk}\right\} - 4kQQ.\end{aligned}$$

Having multiplied this equation by $2s = 2(k-k^3)QQ$ one obtains:

$$\frac{2sd^2s}{dk^2} = 2(k-k^3)Q\frac{dQ}{dk}\left\{2(1-3k^2)QQ + 2(k-k^3)Q\frac{dQ}{dk}\right\} - 8k^2(1-k^2)Q^4,$$

or because it is:

$$\begin{aligned}2(k-k^3)Q\frac{dQ}{dk} &= \frac{ds}{dk} - (1-3k^2)QQ \\ 2(1-3k^2)QQ + 2(k-k^3)Q\frac{dQ}{dk} &= \frac{ds}{dk} + (1-3k^2)QQ,\end{aligned}$$

we obtain:

$$\frac{2sd^2s}{dk^2} = \left[\frac{ds}{dk}\right]^2 - (1-3k^2)^2Q^4 - 8k^2(1-k^2)Q^4 = \left[\frac{ds}{ds}\right]^2 - (1+k^2)^2Q^4,$$

or

$$(9.) \qquad \frac{2sd^2s}{dk^2} - \left[\frac{ds}{dk}\right]^2 + \left[\frac{1+k^2}{k-k^3}\right]^2 ss = 0.$$

But having put $a'K + b'K' = Q'$, $\frac{Q'}{Q} = t$ we see that $\frac{dt}{dk} = \frac{m}{(k-k^3)QQ} = \frac{m}{s}$, where m denotes a constant whence $s = \frac{mdk}{dt}$. Let us transform equation (9.) into another in which dt is put constant. It will be $\frac{ds}{dk} = \frac{md^2k}{dtdk}$, $\frac{d^2}{dk^2} = \frac{md^3k}{dtdk^2} - \frac{md^2k^2}{dtdk^3}$; having substituted these it arises from equation (9.):

$$\frac{2d^3k}{dt^2dk} - \frac{3d^2k^2}{dt^2dk^2} + \left[\frac{1+k^2}{k-k^3}\right]^2\frac{dk^2}{dt^2} = 0,$$

or

$$(10.) \qquad 2d^3kdk - 3d^2k^2 + \left[\frac{1+k^2}{k-k^3}\right]^2 \frac{dk^2}{dk^4} = 0,$$

where is to be differentiated with respect to t which went out of the equation.

By putting $\frac{\alpha'\Lambda+\beta'\Lambda'}{\alpha\Lambda+\beta\Lambda^{\beta}} = \omega$, one will be able to determine constants α, β, α', β', if λ is the transformed modulus, in such a way that $t = \omega$; and in completely similar way we obtain:

$$(11.) \qquad 2d^3\lambda d\lambda - 3d^2\lambda^2 + \left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2 d\lambda^4 = 0,$$

in which equation it is to be differentiated with respect to $t = \omega$. Multiply equation (10.) by $d\lambda^2$, equation (11.) by dk^2; after the substitution one obtains:

$$(12.)\ 2dkd\lambda\left\{d\lambda d^3k - dkd^3\lambda\right\} - 3\left\{d\lambda^2d^2k^2 - dk^2d^2\lambda^2\right\} + dk^2d\lambda^2\left\{\left[\frac{1+k^2}{k-k^3}\right]^2 dk^2 - \left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2 d\lambda^2\right\} = 0.$$

But this equations agrees with equation (8.) in which we know that any preferred differential can be considered as constant and even though it was found having done the substitution that dt is a constant differential it will also hold whatever other is considered as such.

So, glow and behold this differential equation of third order which nevertheless has innumerable particular solutions, nevertheless some particular solutions, of course, are those we called modular equations. But the complete integral depends on elliptic functions which is $t = \omega$ or

$$\frac{a'K + bK'}{aK + bK'} = \frac{\alpha'\Lambda + \beta'\Lambda'}{\alpha\Lambda + \beta\Lambda'},$$

which equation can also be represented this way:

$$mK\Lambda + m'K'\Lambda' + m''K\Lambda' + m'''K'\Lambda = 0,$$

m, m', m'', m''' denoting arbitrary constants. This integration we consider to be of highest depth.

We could inquire whether modular equations for the transformations of third and fifth order indeed, what they have to, satisfy our differential equation of third order. But because this seems to demand too long calculations, it shall suffice to show the same on the transformation of second order, where $\lambda = \frac{1-k'}{1+k'}$.

Let us consider dk' to be constant, it is:

$$\begin{aligned}
\lambda &= \frac{1-k'}{1+k'} = -1 + \frac{2}{1+k'} & k^2 + k'^2 &= 1 \\
\frac{d\lambda}{dk'} &= \frac{-2}{(1+k')^2} & \frac{dk}{dk'} &= -\frac{-k'}{k} \\
\frac{d^2\lambda}{dk'^2} &= \frac{4}{(1+k')^3} & \frac{d^2k}{dk'^2} &= -\frac{-1}{k} - \frac{k'^2}{k^3} = \frac{-1}{k^3} \\
\frac{d^3\lambda}{dk'^3} &= \frac{-12}{(1+k')^3} & \frac{d^3k}{dk'^3} &= -\frac{-3k'}{k^5}
\end{aligned}$$

Hence, it is:

$$\begin{aligned}
\frac{dk^2 d^2\lambda^2 - d\lambda^2 d^2k^2}{dk'^6} &= \frac{16k'^2}{k^2(1+k')^6} - \frac{4}{k^6(1+k')^4} \\
= \frac{4[4k^4k'^2 - (1+k')^2]}{k^6(1+k')^6} &= \frac{4[4k'^2(1-k')-1]}{k^6(1+k')^4}.
\end{aligned}$$

Further, it is obtained:

$$\begin{aligned}
\frac{dkd^3\lambda - d\lambda d^3k}{dk'^4} &= \frac{12k'}{k(1+k')^4} - \frac{6k'}{k^5(1+k')^2} = \frac{6k'[2(1-k')^2-1]}{k^5(1+k')^2} \\
\frac{dkd\lambda[dkd^3\lambda - d\lambda d^3k]}{dk'^6} &= \frac{12k'^2[2(1-k')-1]}{k^6(1+k')^4},
\end{aligned}$$

whence

$$\frac{3[dk^2d^2\lambda^2 - d\lambda^2 d^2k^2] - 2dkd\lambda[dkd^3\lambda - d\lambda d^3k]}{dk'^6} = \frac{12(2k'^2-1)}{k^6(1+k')^4}.$$

Further, it is

$$\left[\frac{1+k}{k-k^3}\right]^2 \frac{dk^2}{dk'^2} = \frac{(1+k'^2)^2}{k^4k'^2}$$

$$\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk'^2}=\frac{4}{(1+k')^4}\left[\frac{1+k'}{1-k'}\right]^2\left[\frac{1+k'^2}{2k'}\right]^2=\frac{(1+k'^2)^2}{k'^2k^4},$$

whence

$$\left[\frac{1+k^2}{k-k^3}\right]^2\frac{dk^2}{dk'^2}-\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk'^2}=\frac{3(1-2k'^2)}{k^4k'^2}$$

$$\frac{dk^2d\lambda^2}{dk'^4}=\left\{\left[\frac{1+k^2}{k-k^3}\right]^2\frac{dk^2}{dk'^2}-\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk'^2}\right\}=\frac{12(1-2k'^2)}{k^6(1+k')^4}.$$

Hence, it finally becomes what is has to be

$$\left.\begin{array}{l}\dfrac{3[dk^2d^2\lambda^2-d\lambda^2d^2k^2]-2dkd\lambda[dkd^3\lambda-d\lambda d^3k]}{dk'^6}\\ +\dfrac{dk^2d\lambda^2}{dk'^4}\left\{\left[\dfrac{1+k^2}{k-k^3}\right]^2-\left[\dfrac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\dfrac{d\lambda}{dk'^2}\right\}\end{array}\right\}=\frac{12(2k'^2-1)}{k^6(1+k')^4}+\frac{12(1-2k'^2)}{k^6(1+k')^4}=0.$$

If there would be a finished theory, if a differential equation has algebraic solutions, to find them all, we could from the differential equation propounded find all modular equations which belong to the single orders of transformation. Nevertheless, I know no one who has tackled this difficult matter worth of the analysts' attention, except Condorcet.

34.

The equation found above

$$MM=\frac{1}{n}\cdot\frac{\lambda(1-\lambda^2)}{k(1-k^2)}\cdot\frac{dk}{d\lambda},$$

by means of which it is possible from the found modular equation to also find the quantity M immediately, seems to be worth to consider it a little bit more. It is not clear on first sight how the values of the quantity M agree with the found equation in the transformations of third and fifth order. Therefore, let us consider this more accurately.

a) In the transformation of *third* order having put $u=\sqrt[4]{k}$, $v=\sqrt[4]{\lambda}$ we find:

$$(1.)\qquad u^4-v^4+2uv(1-u^2v^2)=0,$$

which equation we also exhibited this way (§ 16):

$$(2.)\qquad \left(\frac{v+2u^3}{v}\right)\left(\frac{u-2v^3}{u}\right) = -3.$$

Further, we saw that it is:

$$(3.)\qquad M = \frac{v}{v+2u^3} = \frac{2v^3-u}{3u}.$$

Having differentiated equation (1.) we obtain:

$$\frac{du}{dv} = \frac{2v^3-u+3u^3v^2}{2u^3+v-3u^2v^3},$$

or having put $\left(\frac{v+2u^3}{v}\right)\left(\frac{2v^3-u}{u}\right)$ instead of 3:

$$(4.)\qquad \frac{du}{dv} = \frac{2v^3-u}{2u^3+v}\cdot\frac{1+uv^2u^2+2u^5v}{1+u^2v^2-2uv^5}.$$

From equation (1.) it follows:

$$\begin{aligned}1-u^8 &= (1+u^4)[1-v^4+2uv(1-u^2v^2)]\\ &= 1-u^4v^4+u^4-v^4+2uv(1+u^4)(1-u^2v^2)\\ &= 1-u^4v^4+2u^5v(1-u^2v^2) = (1-u^2v^2)(1+u^2v^2+2u^5v).\end{aligned}$$

The same way one find:

$$1-v^8 = (1-u^2v^2)(1+u^2v^2-2uv^5),$$

whence:

$$\frac{1-v^8}{1-u^8} = \frac{1+u^2v^2-2uv^5}{1+u^2v^2+2u^5v},$$

or from equation (4.):

$$\frac{1-v^8}{1-u^8}\cdot\frac{du}{dv} = \frac{2v^3-u}{2u^3+v}..$$

Having multiplied this equation by:

$$\frac{v}{3u} = \frac{v^2}{(2u^3+v)(2v^3-u)},$$

it arises:

$$\frac{1}{3} \cdot \frac{v(1-v^8)}{u(1-u^8)} \cdot \frac{du}{dv} = \frac{1}{3} \cdot \frac{\lambda(1-\lambda^2)}{k(1-k^2)} \cdot \frac{dk}{d\lambda} = \left[\frac{v}{v+2u^3}\right]^2 = MM,$$

Q.D.E.

b) In the transformation of *fifth* order having put $u = \sqrt[4]{k}$, $v = \sqrt[4]{\lambda}$ we found:

$$(1.) \qquad u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0,$$

which equation we also exhibited in these ways (§§. 16 and 30):

$$(2.) \qquad \frac{u+v^5}{u(1+u^3v)} \cdot \frac{v-u^5}{v(1-uv^3)} = 5$$

$$(3.) \qquad (u^2-v^2)^6 = 16u^2v^2(1-u^8)(1-v^8).$$

Further, we found:

$$(4.) \qquad M = \frac{v(1-uv^3)}{v-u^5} = \frac{u+v^5}{5u(1+u^3v)}.$$

Having differentiated equation (3.) we obtain:

$$6uv(1-u^8)(1-v^8)(udu-vdv) = u(u^2-v^2)(1-u^8)(1-5v^8)dv + v(u^2-v^2)(1-v^8)(1-5u^8)du,$$

Having multiplied equation (1.) multiplied by u^4, v^4 one finds:

$$5u^2 - u^{10} + v^2 - 5u^8v^2 = (1-u^4v^4)(v^2 + 5u^2 + 4u^5v)$$
$$5v^2 - v^{10} + u^2 - 5u^2v^8 = (1-u^4v^4)(u^2 + 5v^2 - 4uv^5),$$

whence equation (5.) goes over into this one:

$$(6.) \qquad \frac{v(1-v^8)}{u(1-u^8)} \cdot \frac{du}{dv} = \frac{u^2+5v^2-4uv^5}{v^2+5u^2+4u^5v}.$$

Put $u+v^5$, $u+u^4v = B$, $v-u^5 = C$, $v-uv^4 = D$ such that:

$$\begin{aligned}
&\frac{AC}{BC} = 5, \quad \text{or} \quad AC = 5BD,\\
&\frac{D}{C} = \frac{A}{5B} = M;\\
&u^2 + 5v^2 - 4uv^5 = uA + 5vD\\
&v^2 + 5u^2 + 4u^5 v = vC + 5uB,
\end{aligned}$$

it will be:

$$\begin{aligned}
(7.) \qquad \frac{v(1-v^8)}{u(1-u^8)} \cdot \frac{du}{dv} &= \frac{uA + 5vD}{vC + 5uB} = \frac{uAB + vAC}{vCD + uAC} \cdot \frac{D}{B}\\
&= \frac{uB + vC}{vD + UA} \cdot \frac{AB}{BC} = \frac{AD}{BC} = 5MM.
\end{aligned}$$

For, it is:

$$uB + vC = vD + uA = u^2 + v^2.$$

Hence, also:

$$MM = \frac{1}{5} \cdot \frac{v(1-v^8)}{u(1-u^8)} \cdot \frac{du}{dv} = \frac{1}{5} \cdot \frac{\lambda(1-\lambda^2)}{k(1-k^2)} \cdot \frac{dk}{d\lambda}.$$

Q.D.E.

2 Theory of the Expansion of Elliptic Functions

35.

Having propounded a real modulus k, smaller than unity, we saw that the modulus

$$\lambda = k^n \left\{ \sin\operatorname{coam} \frac{2K}{n} \sin\operatorname{coam} \frac{4K}{n} \cdots \sin\operatorname{coam} \frac{(n-1)K}{n} \right\}^4$$

into which it is transformed by the first transformation of n-th order, while n grows, rapidly converges to zero, and hence for the limit $n = \infty$ it is $\lambda = 0$. Then, it will be $\Lambda = \frac{\pi}{2}$, $\operatorname{am}(u, \lambda) = u$ whence from the formulas $\Lambda = \frac{K}{nM}$, $\Lambda' = \frac{K'}{M}$ we obtain:

$$nM = \frac{2K}{\pi}, \quad \frac{\Lambda'}{n} = \frac{K'}{nM} = \frac{\pi K'}{2K}.$$

Now, in the formulas for the supplementary transformation of the first in § 27 let us put $\frac{u}{n}$ instead of u, $n = \infty$: $\operatorname{am}\left(\frac{u}{M}, \lambda\right)$ goes over into $\operatorname{am}\left(\frac{u}{nM}, \lambda\right) = \frac{\pi u}{2K}$, $y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)$ into $\sin\frac{\pi u}{2K}$, further $\operatorname{am}(nu)$ into $\operatorname{am}(u)$. Hence, from those formulas we obtain the following:

$$\begin{aligned}
\sin\operatorname{am} u &= \frac{2Ky}{\pi} \cdot \frac{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{2i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{K}}\right)\cdots}{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots} \\
\cos\operatorname{am} u &= \sqrt{1-y^2} \cdot \frac{\left(1 - \frac{y^2}{\cos^2 \frac{i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\cos^2 \frac{2i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\cos^2 \frac{3i\pi K'}{K}}\right)\cdots}{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots} \\
\Delta\operatorname{am} u &= \frac{\left(1 - \frac{y^2}{\cos^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\cos^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots}{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots}
\end{aligned}$$

$$\sqrt{\frac{1-\sin\operatorname{am}u}{1+\sin\operatorname{am}u}} = \sqrt{\frac{1-y}{1+y}}\cdot\frac{\left(1-\frac{y}{\cos\frac{i\pi K'}{K}}\right)\left(1-\frac{y}{\cos\frac{2i\pi K'}{K}}\right)\left(1-\frac{y}{\cos\frac{3i\pi K'}{K}}\right)\cdots}{\left(1+\frac{y}{\cos\frac{i\pi K'}{K}}\right)\left(1+\frac{y}{\cos\frac{2i\pi K'}{K}}\right)\left(1+\frac{y}{\cos\frac{3i\pi K'}{K}}\right)\cdots}$$

$$\sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}} = \frac{\left(1-\frac{y}{\cos\frac{i\pi K'}{2K}}\right)\left(1-\frac{y}{\cos\frac{3i\pi K'}{2K}}\right)\left(1-\frac{y}{\cos\frac{5i\pi K'}{2K}}\right)\cdots}{\left(1+\frac{y}{\cos\frac{i\pi K'}{2K}}\right)\left(1+\frac{y}{\cos\frac{3i\pi K'}{2K}}\right)\left(1+\frac{y}{\cos\frac{5i\pi K'}{2K}}\right)\cdots}$$

$$\sin\operatorname{am}u = -\frac{\pi y}{kK}\cdot\left(\frac{\cos\frac{i\pi K'}{2k}}{\sin^2\frac{i\pi K'}{2K}-y^2}+\frac{\cos\frac{3i\pi K'}{2k}}{\sin^2\frac{3i\pi K'}{2K}-y^2}+\frac{\cos\frac{5i\pi K'}{2k}}{\sin^2\frac{5i\pi K'}{2K}-y^2}+\cdots\right)$$

$$\cos\operatorname{am}u = \frac{i\pi\sqrt{1-y^2}}{kK}\cdot\left(\frac{\sin\frac{i\pi K'}{2k}}{\sin^2\frac{i\pi K'}{2K}-y^2}-\frac{\sin\frac{3i\pi K'}{2k}}{\sin^2\frac{3i\pi K'}{2K}-y^2}+\frac{\sin\frac{5i\pi K'}{2k}}{\sin^2\frac{5i\pi K'}{2K}-y^2}-\cdots\right).$$

In the following, let us put $e^{\frac{-\pi k'}{K}} = q$, $\frac{\pi u}{2K} = x$, or $u = \frac{2Kx}{\pi}$, whence $y = \sin\frac{\pi u}{2K} = \sin x$; it is:

$$\sin\frac{mi\pi K'}{K} = \frac{q^m - q^{-m}}{2i} = \frac{i(1-q^{2m})}{2q^m}$$

$$\cos\frac{mi\pi K'}{K} = \frac{q^m + q^{-m}}{2} = \frac{1+q^{2m})}{2q^m},$$

whence:

$$1-\frac{y^2}{\sin^2\frac{mi\pi K'}{K}} = 1+\frac{4q^{2m}\sin^2 x}{(1-q^{2m})^2} = \frac{1-2q^{2m}\cos 2x+q^{4m}}{(1-q^{2m})^2}$$

$$1-\frac{y^2}{\cos^2\frac{mi\pi K'}{K}} = 1-\frac{4q^{2m}\sin^2 x}{(1+q^{2m})^2} = \frac{1+2q^{2m}\cos 2x+q^{4m}}{(1+q^{2m})^2}$$

$$1\pm\frac{y}{\cos\frac{mi\pi K'}{K}} = 1\pm\frac{2q^m\sin x}{(1+q^{2m})^2} = \frac{1\pm 2q^m\sin x+q^{2m}}{1+q^{2m}}$$

$$\frac{-\cos\frac{mi\pi K'}{K}}{\sin^2\frac{mi\pi K'}{K}-y^2} = \frac{2q^m(1+q^{2m})}{1-2q^{2m}\cos 2x+q^{4m}}$$
$$\frac{i\sin\frac{mi\pi K'}{K}}{\sin^2\frac{mi\pi K'}{K}-y^2} = \frac{2q^m(1-q^{2m})}{1-2q^{2m}\cos 2x+q^{4m}}$$

Having prepared these things and having put for the sake of brevity:

$$A = \left\{\frac{(1-q)}{(1-q^2)}\frac{(1-q^3)}{(1-q^4)}\frac{(1-q^5)\cdots}{(1-q^6)\cdots}\right\}^2$$
$$B = \left\{\frac{(1-q)}{(1+q^2)}\frac{(1-q^3)}{(1+q^4)}\frac{(1-q^5)\cdots}{(1+q^6)\cdots}\right\}^2$$
$$C = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)\cdots}{(1+q^5)\cdots}\right\}^2,$$

the following fundamental expansions of the elliptic functions into infinite products arise:

$$(1.)\quad \sin\operatorname{am}\frac{2Kx}{\pi} = \frac{2AK}{\pi}\sin x\cdot\frac{(1-2q^2\cos 2x+q^4)}{(1-2q\cos 2x+q^2)}\frac{(1-2q^4\cos 2x+q^8)}{(1-2q^3\cos 2x+q^6)}\frac{(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$
$$(2.)\quad \sin\operatorname{am}\frac{2Kx}{\pi} = B\cos x\cdot\frac{(1+2q^2\cos 2x+q^4)}{(1-2q\cos 2x+q^2)}\frac{(1+2q^4\cos 2x+q^8)}{(1-2q^3\cos 2x+q^6)}\frac{(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$
$$(3.)\quad \Delta\operatorname{am}\frac{2Kx}{\pi} = C\cdot\frac{(1+2q\cos 2x+q^2)}{(1-2q\cos 2x+q^2)}\frac{(1+2q^3\cos 2x+q^6)}{(1-2q^3\cos 2x+q^6)}\frac{(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$

$$(4.)\quad \sqrt{\frac{1-\sin\operatorname{am}\frac{2Kx}{\pi}}{1+\sin\operatorname{am}\frac{2Kx}{\pi}}} = \sqrt{\frac{1-\sin x}{1+\sin x}}\cdot\frac{(1-2q\sin x+q^2)}{(1+2q\sin x+q^2)}\frac{(1-2q^2\sin x+q^4)}{(1+2q^2\sin x+q^4)}\frac{(1-2q^3\sin x+q^6)\cdots}{(1+2q^3\sin x+q^6)\cdots}$$
$$(5.)\quad \sqrt{\frac{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}} = \sqrt{\frac{1-\sin x}{1+\sin x}}\cdot\frac{(1-2\sqrt{q}\sin x+q)}{(1+2\sqrt{q}\sin x+q)}\frac{(1-2\sqrt{q^3}\sin x+q^3)}{(1+2\sqrt{q^3}\sin x+q^3)}\frac{(1-2\sqrt{q^5}\sin x+q^5)\cdots}{(1+2\sqrt{q^5}\sin x+q^5)\cdots}$$

and another system of formulas which the resolution into simple fractions yields:

$$(6.)\quad \sin\operatorname{am}\frac{2Kx}{\pi} = \frac{2\pi}{kK}\sin x\left(\frac{\sqrt{q}(1+q)}{1-2q\cos 2x+q^2}+\frac{\sqrt{q^3}(1+q^3)}{1-2q^3\cos 2x+q^6}+\frac{\sqrt{q^5}(1+q^5)}{1-2q^5\cos 2x+q^{10}}+\cdots\right)$$

$$(7.)\quad \cos\operatorname{am}\frac{2Kx}{\pi} = \frac{2\pi}{kK}\cos x\left(\frac{\sqrt{q}(1-q)}{1-2q\cos 2x+q^2}-\frac{\sqrt{q^3}(1-q^3)}{1-2q^3\cos 2x+q^6}+\frac{\sqrt{q^5}(1-q^5)}{1-2q^5\cos 2x+q^{10}}-\cdots\right)$$

To these we add the following from the same source:

$$(8.)\quad 1-\Delta\operatorname{am}\frac{2Kx}{\pi} = \frac{4\pi\sin^2 x}{K}\left(\frac{q\left(\frac{1+q}{1-q}\right)}{1-2q\cos 2x+q^2}-\frac{q^3\left(\frac{1+q^3}{1-q^3}\right)}{1-2q^3\cos 2x+q^6}+\frac{q^5\left(\frac{1+q^5}{1-q^5}\right)}{1-2q^5\cos 2x+q^{10}}-\cdots\right)$$

$$(9.)\quad \operatorname{am}\frac{2Kx}{\pi} = \pm x+2\arctan\frac{(1+q)\tan x}{1-q}-2\arctan\frac{(1+q^3)\tan x}{1-q^3}+2\arctan\frac{(1+q^5)\tan x}{1-q^5}-\cdots$$

In the last formula the upper sign is to be chosen if one stops at a negative term, the upper if one stops at a positive term.

36.

Let us consider the formulas (1.), (2.), (3.) in which especially the values of the quantities we denoted by A, B, C are to be found. They are easily found by putting $x=\frac{\pi}{2}$ from formulas (3.), (1.):

$$k' = C\left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)\cdots}{(1+q^5)\cdots}\right\}^2 = CC,$$

whence

$$1=\frac{2AK}{\pi}\left\{\frac{(1+q^2)}{(1+q)}\frac{(1+q^4)}{(1+q^3)}\frac{(1+q^6)\cdots}{(1+q^5)\cdots}\right\}^2=\frac{2AK}{\pi}\cdot\frac{C}{B}=\frac{2\sqrt{k'}AK}{\pi B},$$

whence

$$B=\frac{2\sqrt{k'}AK}{\pi}.$$

But to find the value of A other tricks are to be used.

Let us put $e^{ix} = U$: If x is changed into $x + \frac{i\pi K'}{2K}$, U goes over into $\sqrt{q}U$, $\sin\operatorname{am}\frac{2Kx}{\pi}$ into

$$\sin\operatorname{am}\left(\frac{2Kx}{\pi} + iK'\right) = \frac{1}{k\sin\operatorname{am}\frac{2Kx}{\pi}}.$$

But from formula (1.) we obtain:

$$\sin\operatorname{am}\frac{2Kx}{\pi} = \frac{AK}{\pi}\left(\frac{U-U^{-1}}{i}\right)\frac{[(1-q^2U^2)(1-q^4U^2)\cdots]}{[(1-qU^2)(1-q^3U^2)\cdots]}\cdots\frac{[(1-q^2U^{-2})(1-q^4U^{-2})\cdots]}{[(1-qU^{-2})(1-q^3U^{-2})\cdots]},$$

whence by changing x to $x + \frac{i\pi K'}{2K}$:

$$\frac{1}{k\sin\operatorname{am}\frac{2Kx}{\pi}} = \frac{AK}{\pi}\left(\frac{\sqrt{q}U - \sqrt{q^{-1}}U^{-1}}{i}\right)\frac{[(1-q^3U^2)(1-q^5U^2)\cdots]}{[(1-^2U^2)(1-q^4U^2)\cdots]}\cdots\frac{[(1-q^2U^{-2})(1-U^{-2})\cdots]}{[(1-q^3U^{-2})(1-q^2U^{-2})\cdots]},$$

having multiplied those by each other, because it is:

$$\frac{\sqrt{q}U - \sqrt{q^{-1}}U^{-1}}{1-U^{-2}} = -\frac{1}{\sqrt{q}}\cdot\frac{1-qU^2}{U-U^{-1}},$$

it arises:

$$\frac{1}{k} = \frac{1}{\sqrt{q}}\left(\frac{AK}{\pi}\right)^2, \quad \text{or} \quad A = \frac{\pi\sqrt[4]{q}}{\sqrt{k}K}; \quad \text{hence} \quad \frac{2KA}{\pi} = \frac{2\sqrt[4]{q}}{\sqrt{k}}.$$

Hence, it will be $B = \frac{2\sqrt{k'}AK}{\pi} = 2\sqrt[4]{q}\sqrt{\frac{k'}{k}}$. Therefore, it is:

$$\begin{aligned}
\sin\operatorname{am}\frac{2Kx}{\pi} &= \frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}\\
\cos\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{\frac{k'}{k}}\cdot\frac{2\sqrt[4]{q}\cos x(1+2q^2\cos 2x+q^4)(1+2q^4\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}\\
\Delta\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{k'}\cdot\frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}.
\end{aligned}$$

Having multiplied these equations by each other:

$$\begin{aligned}
B = 2\sqrt[4]{q}\sqrt{\frac{k'}{k}} &= \left\{\frac{(1-q)}{(1+q^2)}\frac{(1-q^3)}{(1+q^4)}\frac{(1-q^5)\cdots}{(1+q^6)\cdots}\right\}^2\\
C = \sqrt{k'} &= \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^2)}\frac{(1-q^5)\cdots}{(1+q^3)\cdots}\right\}^2,
\end{aligned}$$

it arises:

$$\frac{2\sqrt[4]{q}k'}{\sqrt{k}} = \frac{[(1-q)(1-q^3)(1-q^5)\cdots]^4}{(1+q)(1+q^2)(1+q^3)\cdots]^2}.$$

But according to Euler in *Introductio (de Partitione Numerorum)* it is:

$$\begin{aligned}(1+q)(1+q^2)(1+q^3)\cdots &= \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots},\end{aligned}$$

whence we obtain:

$$(1.) \qquad [(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots]^6 = \frac{2\sqrt[4]{q}k'}{\sqrt{k}}.$$

Recalling the formula:

$$A = \frac{\pi\sqrt[4]{q}}{\sqrt{k}K} = \left\{\frac{(1-q)(1-q^3)(1-q^5)\cdots}{(1-q^2)(1-q^4)(1-q^6)\cdots}\right\}^2,$$

it is:

$$(2.) \qquad [(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots]^6 = \frac{2kk'K^3}{\pi^3\sqrt{q}},$$

whence also:

$$(3.) \qquad [(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots]^6 = \frac{4\sqrt{k}k'k'K^3}{\pi^3\sqrt[4]{q}}.$$

One can add these formulas which easily follow:

$$\begin{aligned}(4.) \qquad [(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots]^6 &= \frac{2\sqrt[4]{q}}{\sqrt{kk'}} \\ (5.) \qquad [(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots]^6 &= \frac{k}{4\sqrt{k'}\sqrt{q}} \\ (6.) \qquad [(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots]^6 &= \frac{\sqrt{k}}{2k'\sqrt[4]{q}}.\end{aligned}$$

From these one also concludes:

$$
\begin{aligned}
&(7.) && k && = 4\sqrt{q}\left\{\frac{(1+q^2)}{(1+q)}\frac{(1+q^4)}{(1+q^3)}\frac{(1+q^6)\cdots}{(1+q^5)\cdots}\right\}^4 \\
&(8.) && k' && = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)\cdots}{(1+q^5)\cdots}\right\}^4 \\
&(9.) && \frac{2K}{\pi} && = \left\{\frac{(1-q^2)}{(1-q)}\frac{(1-q^4)}{(1-q^3)}\frac{(1-q^6)\cdots}{(1-q^5)\cdots}\right\}^2\left\{\frac{(1+q)}{(1+q^2)}\frac{(1+q^3)}{(1+q^4)}\frac{(1+q^5)\cdots}{(1+q^6)\cdots}\right\}^2 \\
&(10.) && \frac{2kK}{\pi} && = 4\sqrt{q}\left\{\frac{(1-q^4)}{(1-q^2)}\frac{(1-q^8)}{(1-q^6)}\frac{(1-q^{12})\cdots}{(1-q^{10})\cdots}\right\}^2 \\
&(11.) && \frac{2k'K}{\pi} && = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^3)\cdots}{(1+q^3)\cdots}\right\}^2 \\
&(12.) && \frac{2\sqrt{k}K}{\pi} && = 2\sqrt[4]{q}\left\{\frac{(1-q^2)}{(1-q)}\frac{(1-q^4)}{(1-q^3)}\frac{(1-q^6)\cdots}{(1-q^5)\cdots}\right\}^2 \\
&(13.) && \frac{2\sqrt{k'}K}{\pi} && = \left\{\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^4)}{(1+q^4)}\frac{(1-q^6)\cdots}{(1+q^6)\cdots}\right\}^2.
\end{aligned}
$$

From formulas (7.), (8.) follows this non-obvious identity:

$$(14.)\quad [(1-q)(1-q^3)(1-q^5)\cdots]^8+16q[(1+q^2)(1+q^4)(1+q^6)\cdots]^8=[(1+q)(1+q^3)(1+q^5)\cdots]^8.$$

37.

We have seen above, where the properties of the modular equations were discussed, having changed k into to $\frac{1}{k}$, that k goes over into $k(K-iK')$, K' into kK'; further that it is:

$$
\begin{aligned}
\sin\operatorname{am}\left(ku,\frac{ik'}{k}\right) &= \cos\operatorname{coam}(u,k') \\
\cos\operatorname{am}\left(ku,\frac{ik'}{k}\right) &= \sin\operatorname{coam}(u,k') \\
\Delta\operatorname{am}\left(ku,\frac{ik'}{k}\right) &= \frac{1}{\Delta\operatorname{am}(u,k')}.
\end{aligned}
$$

Having interchanged k and k', it follows from this, if k' goes over into $\frac{1}{k'}$ or k into $\frac{ik}{k'}$, that at the same time K goes over into $k'K$, K' into $k'(K' - iK)$; further, that it is:

$$\begin{aligned} \sin\operatorname{am}\left(k'u, \frac{ik}{k'}\right) &= \cos\operatorname{coam} u \\ \cos\operatorname{am}\left(k'u, \frac{ik}{k'}\right) &= \sin\operatorname{coam} u \\ \Delta\operatorname{am}\left(k'u, \frac{ik}{k'}\right) &= \frac{1}{\Delta\operatorname{am} u}, \end{aligned}$$

whence also:

$$\operatorname{am}\left(k'u, \frac{ik}{k'}\right) = \frac{\pi}{2} - \operatorname{coam} u.$$

But, having changed K into $k'K$, K' into $k'(K' - iK)$, $q = e^{\frac{\pi K'}{K}}$ goes over into $-q$, whence vice versa follows:

Theorem I

Having changed q into $-q$, we have:

$$\begin{array}{llllll} k & \text{goes over into} & \frac{ik}{k'}, & k' & \text{goes over into} & \frac{1}{k'} \\ K & \text{goes over into} & k'K, & K' & \text{goes over into} & k'(K' - iK) \end{array}$$

$$\begin{array}{lll} \sin\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \cos\operatorname{coam}\frac{2Kx}{\pi} \\ \cos\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \sin\operatorname{coam}\frac{2Kx}{\pi} \\ \Delta\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \frac{1}{\Delta\operatorname{am}\frac{2Kx}{\pi}} \\ \operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \frac{\pi}{2} - \operatorname{coam}\frac{2Kx}{\pi}; \end{array}$$

having changed q to $-q$ and x to $x - \frac{\pi}{2}$

$$\begin{array}{lll}
\operatorname{am}\dfrac{2Kx}{\pi} & \text{goes over into} & \dfrac{\pi}{2} - \operatorname{am}\dfrac{2Kx}{\pi} \\
\sin\operatorname{am}\dfrac{2Kx}{\pi} & \text{goes over into} & \cos\operatorname{am}\dfrac{2Kx}{\pi} \\
\cos\operatorname{am}\dfrac{2Kx}{\pi} & \text{goes over into} & \sin\operatorname{am}\dfrac{2Kx}{\pi} \\
\Delta\operatorname{am}\dfrac{2Kx}{\pi} & \text{goes over into} & \dfrac{1}{k'}\Delta\operatorname{am}\dfrac{2Kx}{\pi}.
\end{array}$$

At last, let us investigate how the elliptic functions transform having changed q either to q^2 or to $\sqrt{q}$.

We saw above that the modulus λ derived by means of the first real transformation of n-th order from the modulus k enjoys the extraordinary property that it is:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K};$$

hence, having changed k into λ, $q = e^{\frac{-\pi K'}{K}}$ goes over into q^n. The same, proved by us on the transformations of odd order in general, was proved already by Legendre on the transformation of second order long time ago, of course, having put $\lambda = \frac{1-k'}{1+k'}$, that it is:

$$\Lambda = \frac{1+k'}{2}K, \quad \Lambda' = (1+k')K', \quad \frac{\Lambda'}{\Lambda} = 2\frac{K'}{K},$$

whence we see, having changed k into $\frac{1-k'}{1+k'}$, that q goes over to q^2. Hence, we vice versa obtain

Theorem II.

Having changed q into q^2, k goes over into $\frac{1-k'}{1+k'}$, K into $\frac{1+k'}{2}K$, whence also:

k'	goes over into	$\dfrac{2\sqrt{k'}}{1+k'}$	$1+k$	goes over into	$\dfrac{2}{1+k'}$
$k'K$	goes over into	$\sqrt{k'}K$	$1-k$	goes over into	$\dfrac{2k'}{1+k'}$
$\sqrt{k}$	goes over into	$\dfrac{k}{1+k'}$	$1+k'$	goes over into	$\dfrac{(1+\sqrt{k})^2}{1+k'}$
$\sqrt{k}K$	goes over into	$\dfrac{kK}{2}$	$1-k'$	goes over into	$\dfrac{(1+\sqrt{k'})^2}{1+k'}$

From the inversion of this theorem one obtains another

Theorem III.

Having changed q into $\sqrt{q}$, k goes over into $\frac{2\sqrt{k}}{1+k}$, K into $(1+k)K$, whence also:

k'	goes over into	$\dfrac{1-k}{1+k}$	$1+k$	goes over into	$\dfrac{(1+\sqrt{k})^2}{1+k}$
$\sqrt{k'}$	goes over into	$\dfrac{k'}{1+k}$	$1-k$	goes over into	$\dfrac{(1-\sqrt{k})^2}{1+k}$
kK	goes over into	$2\sqrt{k}K$	$1+k'$	goes over into	$\dfrac{2}{1+k}$
$\sqrt{k}K$	goes over into	$k'K$	$1-k'$	goes over into	$\dfrac{2}{1+k}$

These three theorems are confirmed by the expansions propounded in § 35 and § 36 in many ways and they find a very frequent use in the following, by means of which either it is possible to derive even more formulas from others or formulas found from other source are confirmed.

38.

The quantities into which, after having put q^r instead of q, k, k', K go over, we want to denote by $k^{(r)}$, $k^{(r)\prime}$, $K^{(r)}$ so that $k^{(r)}$ is the modulus found by the first real transformation of r-th order and $k^{(r)\prime}$ its complement. Let us put in the equation:

$$\sqrt{k'}\left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)}{(1+q^5)}\frac{(1-q^7)\cdots}{(1+q^7)\cdots}\right\}^2$$

instead of q successively q^2, q^4, q^8, q^{16}, etc. multiplying all equations it arises:

$$\sqrt{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}k^{(16)\prime}\dots} = \left\{\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^4)}{(1+q^4)}\frac{(1-q^6)}{(1+q^6)}\frac{(1-q^8)\cdots}{(1+q^8)\cdots}\right\}^2;$$

but we found:

$$\left\{\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^4)}{(1+q^4)}\frac{(1-q^6)}{(1+q^6)}\frac{(1-q^8)\cdots}{(1+q^8)\cdots}\right\}^2 = \frac{2\sqrt{k'}K}{\pi},$$

whence:

$$(1.) \qquad \frac{2K}{\pi} = \sqrt{\frac{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}k^{(16)\prime}\dots}{k'}}.$$

Because it is $k^{(2)\prime} = \frac{2\sqrt{k'}}{1+k'}$, it becomes from (1.):

$$\left(\frac{2K}{\pi}\right)^2 = \frac{1}{k'}\cdot\frac{2\sqrt{k'}}{1+k'}\cdot\frac{2\sqrt{k^{(2)\prime}}}{1+k^{(2)\prime}}\cdot\frac{2\sqrt{k^{(4)\prime}}}{1+k^{(4)\prime}}\cdot\frac{2\sqrt{k^{(8)\prime}}}{1+k^{(8)\prime}}\cdots,$$

whence having divided by (1.):

$$(2.) \qquad \frac{2K}{\pi} = \frac{2}{1+k'}\cdot\frac{2}{1+k^{(2)\prime}}\cdot\frac{2}{1+k^{(4)\prime}}\cdot\frac{2}{1+k^{(8)\prime}}\cdots$$

This formula is also obtained because it holds:

$$\begin{aligned}\frac{2K}{\pi} &= \frac{2K^{(2)}}{\pi}\cdot\frac{2}{1+k'}\\ \frac{2K^{(2)}}{\pi} &= \frac{2K^{(4)}}{\pi}\cdot\frac{2}{1+k^{(2)\prime}}\\ \frac{2K^{(4)}}{\pi} &= \frac{2K^{(8)}}{\pi}\cdot\frac{2}{1+k^{(4)\prime}}\\ &\cdots,\end{aligned}$$

whence, because as r increases to infinity the limit of the expression $\frac{2K^{(r)}}{\pi}$ is 1, having expanded the infinite product, (2.) arises. Having put:

$$\begin{aligned}
m &= 1, & n &= k' \\
m' &= \frac{m+n}{2}, & n' &= \sqrt{nm} \\
m'' &= \frac{m'+n'}{2}, & n'' &= \sqrt{n'm'} \\
m''' &= \frac{m''+n''}{2}, & n''' &= \sqrt{n''m''}
\end{aligned}$$

it is:

$$\begin{aligned}
k^{(2)\prime} &= \frac{2\sqrt{k'}}{1+k'} &= \frac{n'}{m'} \\
k^{(4)\prime} &= \frac{2\sqrt{(2)k'}}{1+k^{(2)\prime}} &= \frac{n''}{m''} \\
k^{(8)\prime} &= \frac{2\sqrt{(4)k'}}{1+k^{(4)\prime}} &= \frac{n'''}{m'''} \\
&\cdots,
\end{aligned}$$

whence:

$$\frac{2}{1+k'} = \frac{m}{m'}, \quad \frac{2}{1+k^{(2)\prime}} = \frac{m'}{m''}, \quad \frac{2}{1+k^{(4)\prime}} = \frac{m''}{m'''}, \cdots$$

and hence:

$$\frac{2K}{\pi} = \frac{m}{m'} \cdot \frac{m'}{m''} \cdot \frac{m''}{m'''} \cdot \frac{m'''}{m''''} \cdots$$

or while μ denotes the common limit to which $m^{(p)}$, $n^{(p)}$ converge as p grows to infinity:

$$(3.) \qquad \frac{2K}{\pi} = \frac{1}{\mu},$$

These are known from other sources.

Let us now again put in formula:

$$\Delta\,\mathrm{am}\,\frac{2Kx}{\pi} = \sqrt{k'}\frac{(1+2q\cos 2x+q^2)}{(1-2q\cos 2x+q^2)}\cdot\frac{(1+2q^3\cos 2x+q^6)}{(1-2q^3\cos 2x+q^6)}\cdot\frac{(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$

instead of q successively put q^2, q^4, q^8, etc.; further let:

$$S = \Delta\,\mathrm{am}\left(\frac{2K^{(2)}x}{\pi}, k^{(2)}\right)\Delta\,\mathrm{am}\left(\frac{2K^{(4)}x}{\pi}, k^{(4)}\right)\Delta\,\mathrm{am}\left(\frac{2K^{(8)}x}{\pi}, k^{(8)}\right)\cdots$$

Having constructed the infinite product, because it is:

$$\frac{2\sqrt{k'}K}{\pi} = \sqrt{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}k^{(16)\prime}\cdots},$$

we obtain:

$$S = \frac{2\sqrt{k'}K}{\pi}\cdot\frac{(1+2q^2\cos 2x+q^4)}{(1-2q^2\cos 2x+q^4)}\cdot\frac{(1+2q^4\cos 2x+q^8)}{(1-2q^4\cos 2x+q^8)}\cdot\frac{(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q^6\cos 2x+q^{12})\cdots}$$

But from the formulas:

$$\sin\mathrm{am}\,\frac{2Kx}{\pi} = \frac{2}{\sqrt{k}}\cdot\frac{\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}$$

$$\cos\mathrm{am}\,\frac{2Kx}{\pi} = 2\sqrt{\frac{k'}{k}}\cdot\frac{\sqrt[4]{q}\cos x(1+2q^2\cos 2x+q^4)(1+2q^4\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}$$

we obtain:

$$\Delta\,\mathrm{am}\,\frac{2Kx}{\pi} = \frac{1}{\sqrt{k'}}\cdot\frac{\tan x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1+2q\cos 2x+q^4)(1+2q^3\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\cdots}$$

whence this memorable formula arises:

$$(4.)\qquad \tan x = \frac{S\cdot\tan\mathrm{am}\,\frac{2Kx}{\pi}}{\frac{2K}{\pi}}.$$

To demonstrate the same by means of known formulas let us recall a formula for the transformations of second order which Gauss exhibited in the treatise called: *"Determinatio Attractionis"* etc.:

$$\sin\operatorname{am}\frac{2Kx}{\pi}=\frac{(1+k^{(2)})\sin\operatorname{am}\left(\frac{2K^{(2)}x}{\pi},k^{(2)}\right)}{1+k^{(2)}\sin\operatorname{am}\left(\frac{2K^{(2)}x}{\pi},k^{(2)}\right)},$$

which having put for the sake of brevity:

$$\operatorname{am}\left(\frac{2K^{(r)}x}{\pi},k^{(r)}\right)=\varphi^{(r)},\quad \Delta\operatorname{am}\left(\frac{2K^{(r)}x}{\pi},k^{(r)}\right)=\Delta^{(r)},$$

it is exhibited as this:

$$\sin\varphi=\frac{(1+k^{(2)})\sin\varphi^{(2)}}{1+k^{(2)}\sin\varphi^{(2)}},$$

whence also:

$$\begin{aligned}\cos\varphi&=\frac{\cos\varphi^{(2)}\Delta^{(2)}}{1+k^{(2)}\sin^2\varphi^{(2)}}\\ \Delta\varphi&=\frac{1-k^{(2)}\sin^2\varphi^{(2)}}{1+k^{(2)}\sin^2\varphi^{(2)}}\\ \tan\varphi&=\frac{(1+k^{(2)})\tan\varphi^{(2)}}{\Delta^{(2)}}\end{aligned}$$

The last formula can also be represented this way:

$$\frac{\tan\varphi}{\frac{2K}{\pi}}=\frac{\tan\varphi^{(2)}}{\frac{2K^{(2)}}{\pi}}\cdot\frac{1}{\Delta^{(2)}},$$

whence having successively put q^2, q^4, q^8, $\cdots$ instead of q, having done which k, K, φ go over into $k^{(2)}$, $k^{(4)}$, $k^{(8)}$, $\cdots$; $K^{(2)}$, $K^{(4)}$, $K^{(8)}$, $\cdots$; $\varphi^{(2)}$, $\varphi^{(4)}$, $\varphi^{(8)}$, $\cdots$, we obtain:

$$\frac{\tan\varphi^{(2)}}{\frac{2K^{(2)}}{\pi}} = \frac{\tan\varphi^{(4)}}{\frac{2K^{(4)}}{\pi}} \cdot \frac{1}{\Delta^{(4)}}$$
$$\frac{\tan\varphi^{(4)}}{\frac{2K^{(4)}}{\pi}} = \frac{\tan\varphi^{(8)}}{\frac{2K^{(8)}}{\pi}} \cdot \frac{1}{\Delta^{(8)}}$$
$$\frac{\tan\varphi^{(8)}}{\frac{2K^{(8)}}{\pi}} = \frac{\tan\varphi^{(16)}}{\frac{2K^{(16)}}{\pi}} \cdot \frac{1}{\Delta^{(16)}}$$
$$\cdots$$

Now, the limit of the expression

$$\frac{\tan\varphi^{(p)}}{\frac{2K^{(p)}}{\pi}} = \frac{\tan\operatorname{am}\left(\frac{2K^{(p)}x}{\pi}, k^{(p)}\right)}{\frac{2K^{(p)}}{\pi}},$$

as p grows to infinity, is:

$$\tan x;$$

for, then it is $k^{(p)} = 0$, $K^{(p)} = \frac{\pi}{2}$, $\operatorname{am}(u, k^{(p)}) = u$; hence, having taken the infinite product and having, as above, put $S = \Delta^{(2)}\Delta^{(4)}\Delta^{(8)}\cdots$, it arises:

$$\frac{\tan\varphi}{\frac{2k}{\pi}} = \frac{\tan x}{S},$$

which is the formula to be demonstrated.

From the formula:

$$\tan x = \frac{S\cdot\tan\varphi}{\frac{2K}{\pi}}$$

an elegant algorithm for the computation of *indefinite* elliptic integrals of the first kind can be derived; and this by means the easy to prove formula:

$$\Delta^{(2)} = \sqrt{\frac{2(\Delta + k')}{(1+k')(1+\Delta)}}$$

For this aim, we propose the following

Theorem

Having put:

$$\int_0^\varphi \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}} = \Phi$$
$$\sqrt{mm\cos^2\varphi + nn\sin^2\varphi} = \Delta,$$

form the expressions:

$$\begin{array}{llllll}
\dfrac{m+n}{2} & = m' & \sqrt{mn} & = n' & \Delta' & = \sqrt{\dfrac{mm'(\Delta+n)}{m+\Delta}} \\
\dfrac{m'+n'}{2} & = m'' & \sqrt{m'n'} & = n'' & \Delta'' & = \sqrt{\dfrac{m'm''(\Delta'+n')}{m'+\Delta'}} \\
\dfrac{m''+n''}{2} & = m''' & \sqrt{m''n''} & = n''' & \Delta''' & = \sqrt{\dfrac{m''m'''(\Delta''+n'')}{m''+\Delta''}} \\
 & \cdots & & \cdots & & \cdots;
\end{array}$$

μ denoting the common limit to which the quantities $m^{(p)}$, $\Delta^{(p)}$, $n^{(p)}$ as p increases very rapidly converge, it will be:

$$\tan\mu\Phi = \frac{\Delta'\Delta''\Delta'''\cdots}{mm'm''\cdots}\cdot\tan\varphi.$$

By the same methods we uses in the preceding one also finds the value of the infinite product:

$$\frac{2\sqrt[4]{q}}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q^2}}{\sqrt{k^{(2)}}}\cdot\frac{2\sqrt[4]{q^4}}{\sqrt{k^{(4)}}}\cdot\frac{2\sqrt[4]{q^8}}{\sqrt{k^{(8)}}}\cdots$$

For this aim, recall the formulas from § 36 (4.), (5.):

$$[(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots]^6 = \frac{2\sqrt[4]{q}}{\sqrt{kk'}}$$
$$[(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots]^6 = \frac{k}{4\sqrt{k'}\sqrt{q}},$$

the latter of which arise from the first having successively put q^2, q^4, q^8 etc. instead of q and taken the infinite product, whence we obtain:

$$\frac{k}{4\sqrt{k'}\sqrt{q}} = \frac{2\sqrt[4]{q^2}}{\sqrt{k^{(2)}k^{(2)\prime}}} \cdot \frac{2\sqrt[4]{q^4}}{\sqrt{k^{(4)}k^{(4)\prime}}} \cdot \frac{2\sqrt[4]{q^8}}{\sqrt{k^{(8)}k^{(8)\prime}}} \cdots$$

But we already found (1.):

$$\frac{2K}{\pi} = \sqrt{\frac{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}\ldots}{k'}},$$

whence

$$(5.) \qquad \frac{\sqrt{k}}{2\sqrt[4]{q}} \cdot \frac{2K}{\pi} = \frac{2\sqrt[4]{q}}{\sqrt{k}} \cdot \frac{2\sqrt[4]{q^2}}{\sqrt{k^{(2)}}} \cdot \frac{2\sqrt[4]{q^4}}{\sqrt{k^{(4)}}} \cdot \frac{2\sqrt[4]{q^8}}{\sqrt{k^{(8)}}} \cdots$$

These things might seem not to be close to our actual subject; but because they are elegant and very helpful to understand the nature of the propounded expansion, it is useful to have explained them.

2.1 Expansion of elliptic Functions into series of Sines or Cosines of multiples of the Argument

39.

From the formulas given above:

$$(1.) \ \sin\operatorname{am}\frac{2Kx}{\pi} = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin x \cdot \frac{(1-2q^2\cos 2x+q^4)\,(1-2q^4\cos 2x+q^8)\,(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)\,(1-2q^3\cos 2x+q^6)\,(1-2q^5\cos 2x+q^{10})\cdots}$$

$$(2.) \ \cos\operatorname{am}\frac{2Kx}{\pi} = \frac{2\sqrt[4]{q}\sqrt{k'}}{\sqrt{k}} \cos x \cdot \frac{(1+2q^2\cos 2x+q^4)\,(1+2q^4\cos 2x+q^8)\,(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)\,(1-2q^3\cos 2x+q^6)\,(1-2q^5\cos 2x+q^{10})\cdots}$$

$$(3.) \ \Delta\operatorname{am}\frac{2Kx}{\pi} = \sqrt{k'} \cdot \frac{(1+2q\cos 2x+q^2)\,(1+2q^3\cos 2x+q^6)\,(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)\,(1-2q^3\cos 2x+q^6)\,(1-2q^5\cos 2x+q^{10})\cdots}$$

And

$$(4.) \ \sqrt{\frac{1-\sin\operatorname{am}\frac{2Kx}{\pi}}{1+\sin\operatorname{am}\frac{2Kx}{\pi}}} = \sqrt{\frac{1-\sin x}{1+\sin x}} \cdot \frac{(1-2q\sin x+q^2)\,(1-2q^2\sin x+q^4)\,(1-2q^3\sin x+q^6)\cdots}{(1+2q\sin x+q^2)\,(1+2q^2\sin x+q^4)\,(1+2q^3\sin x+q^6)\cdots}$$

$$(5.) \ \sqrt{\frac{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}} = \frac{(1-2\sqrt{q}\sin x+q)\,(1-2\sqrt{q^3}\sin x+q^3)\,(1-2\sqrt{q^5}\sin x+q^5)\cdots}{(1+2\sqrt{q}\sin x+q)\,(1+2\sqrt{q^3}\sin x+q^3)\,(1+2\sqrt{q^5}\sin x+q^5)\cdots},$$

having expanded the logarithms of the single products on the one side of the equations, after obvious reductions, these follow:

$$
\begin{aligned}
&(6.)\quad \ln\sin\operatorname{am}\frac{2Kx}{\pi} = \ln\left\{\frac{2\sqrt[4]{q}}{\sqrt{k}}\sin x\right\} + \frac{2q\cos 2x}{1+q} + \frac{2q^2\cos 4x}{2(1+q^2)} + \frac{2q^3\cos 6x}{3(1+q^3)} + \cdots \\
&(7.)\quad \ln\cos\operatorname{am}\frac{2Kx}{\pi} = \ln\left\{2\sqrt[4]{q}\sqrt{\frac{k'}{k}}\cos x\right\} + \frac{2q\cos 2x}{1-q} + \frac{2q^2\cos 4x}{2(1+q^2)} + \frac{2q^3\cos 6x}{3(1-q^3)} + \cdots \\
&(8.)\quad \ln\Delta\operatorname{am}\frac{2Kx}{\pi} = \ln\sqrt{k'} + \frac{4q\cos 2x}{1-q^2} + \frac{4q^3\cos 4x}{3(1-q^6)} + \frac{4q^5\cos 10x}{5(1-q^{10})} + \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&(9.)\quad \ln\sqrt{\frac{1+\sin\operatorname{am}\frac{2Kx}{\pi}}{1+-\sin\operatorname{am}\frac{2Kx}{\pi}}} = \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + \frac{4q\sin x}{1-q} - \frac{4q^3\sin 3x}{3(1-q^3)} + \frac{4q^5\sin 5x}{5(1-q^5)} - \cdots \\
&(10.)\quad \ln\sqrt{\frac{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}} = \frac{4\sqrt{q}\sin x}{1-q} - \frac{4\sqrt{q^3}\sin 3x}{3(1-q^3)} + \frac{4\sqrt{q^5}\sin 5x}{5(1-q^5)} - \cdots
\end{aligned}
$$

Having differentiated these formulas, if we note the following easy to prove differential formulas:

$$
\begin{aligned}
\frac{d\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx} &= \frac{2k'K}{\pi}\cdot\frac{\cos\operatorname{am}\frac{2Kx}{\pi}}{\cos\operatorname{coam}\frac{2Kx}{\pi}} \\
-\frac{d\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx} &= \frac{2K}{\pi}\cdot\frac{\sin\operatorname{am}\frac{2Kx}{\pi}}{\sin\operatorname{coam}\frac{2Kx}{\pi}} = \frac{2K}{\pi}\cdot\tan\frac{1}{2}\operatorname{am}\frac{4Kx}{\pi} \\
-\frac{d\ln\Delta\operatorname{am}\frac{2Kx}{\pi}}{dx} &= \frac{2k^2K}{\pi}\cdot\sin\operatorname{am}\frac{2Kx}{\pi}\sin\operatorname{coam}\frac{2Kx}{\pi}
\end{aligned}
$$

and

$$\frac{d\ln\sqrt{\frac{1+\sin\operatorname{am}\frac{2Kx}{\pi}}{1-\sin\operatorname{am}\frac{2Kx}{\pi}}}}{dx}=\frac{2K}{\pi}\cdot\frac{1}{\sin\operatorname{coam}\frac{2Kx}{\pi}}$$

$$\frac{d\ln\sqrt{\frac{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}}}{dx}=\frac{2kK}{\pi}\cdot\sin\operatorname{coam}\frac{2Kx}{\pi}$$

we find the following:

$$\text{(11.)}\quad \frac{2k'K}{\pi}\cdot\frac{\cos\operatorname{am}\frac{2Kx}{\pi}}{\cos\operatorname{coam}\frac{2Kx}{\pi}}=\cot x-\frac{4q\sin 2x}{1+q}-\frac{4q^2\sin 4x}{1+q^2}-\frac{4q^3\sin 6x}{1+q^3}-\cdots$$

$$\text{(12.)}\quad \frac{2K}{\pi}\cdot\frac{\sin\operatorname{am}\frac{2Kx}{\pi}}{\sin\operatorname{coam}\frac{2Kx}{\pi}}=\tan x+\frac{4q\sin 2x}{1-q}+\frac{4q^2\sin 4x}{1+q^2}+\frac{4q^3\sin 6x}{1-q^3}+\cdots$$

$$\text{(13.)}\quad \frac{2k^2K}{\pi}\cdot\sin\operatorname{am}\frac{2Kx}{\pi}\sin\operatorname{coam}\frac{2Kx}{\pi}=\frac{8q\sin 2x}{1-q^2}+\frac{8q^3\sin 6x}{1-q^6}-\frac{8q^5\sin 10x}{1-q^{10}}-\cdots$$

$$\text{(14.)}\quad \frac{2K}{\pi\sin\operatorname{coam}\frac{2Kx}{\pi}}=\frac{1}{\cos x}+\frac{4q\cos x}{1-q}-\frac{4q^3\cos 3x}{1-q^3}-\frac{4q^5\cos 5x}{1-q^5}-\cdots$$

$$\text{(15.)}\quad \frac{2kK}{\pi}\cdot\sin\operatorname{coam}\frac{2Kx}{\pi}=\frac{4\sqrt{q}\cos x}{1-q}-\frac{4\sqrt{q^3}\cos 3x}{1-q^3}+\frac{4\sqrt{q^5}\cos 5x}{1-q^5}-\cdots$$

If in these formulas instead of x one puts $\frac{\pi}{2}-x$, it is found:

$$\text{(16.)}\quad \frac{2k'K}{\pi}\cdot\frac{\cos\operatorname{coam}\frac{2Kx}{\pi}}{\cos\operatorname{am}\frac{2Kx}{\pi}}=\tan x-\frac{4q^2\sin 2x}{1+q}+\frac{4q^2\sin 4x}{1+q^2}-\frac{4q^3\sin 6x}{1+q^3}+\cdots$$

$$\text{(17.)}\quad \frac{2K}{\pi}\cdot\frac{\sin\operatorname{coam}\frac{2Kx}{\pi}}{\sin\operatorname{am}\frac{2Kx}{\pi}}=\cot x+\frac{4q^2\sin 2x}{1-q}-\frac{4q^2\sin 4x}{1+q^2}+\frac{4q^3\sin 6x}{1-q^3}-\cdots$$

$$\text{(18.)}\quad \frac{2K}{\pi\sin\operatorname{am}\frac{2Kx}{\pi}}=\frac{1}{\sin x}-\frac{4q\sin x}{1-q}+\frac{4q^3\sin 3x}{1-q^3}+\frac{4q^5\sin 5x}{1-q^5}+\cdots$$

$$\text{(19.)}\quad \frac{2kK}{\pi}\cdot\sin\operatorname{am}\frac{2Kx}{\pi}=\frac{4\sqrt{q}\sin x}{1-q}+\frac{4\sqrt{q^3}\sin 3x}{1-q^3}+\frac{4\sqrt{q^5}\sin 5x}{1-q^5}+\cdots$$

Formula (13.), by putting $\frac{\pi}{2} - x$ instead of x, remains unchanged.

By changing q into -1, from theorem I. § 37 the formulas (11.), (12.) go over into (17.), (16.); (13.) remains unchanged; from the formulas (14.), (15.), (18.), (19.) we obtain:

$$
\begin{aligned}
&(20.) \quad \frac{2k'K}{\pi \cos \operatorname{am} \frac{2Kx}{\pi}} &&= \frac{1}{\cos x} - \frac{4q \cos x}{1+q} + \frac{4q^3 \cos 3x}{1+q^3} - \frac{4q^5 \cos 5x}{1+q^5} + \cdots \\
&(21.) \quad \frac{2kK}{\pi} \cdot \cos \operatorname{am} \frac{2Kx}{\pi} &&= \frac{4\sqrt{q} \cos x}{1+q} + \frac{4\sqrt{q^3} \cos 3x}{1+q^3} + \frac{4\sqrt{q^5} \cos 5x}{1+q^5} + \cdots \\
&(22.) \quad \frac{2k'K}{\pi \cos \operatorname{coam} \frac{2Kx}{\pi}} &&= \frac{1}{\sin x} - \frac{4q \sin x}{1+q} - \frac{4q^3 \sin 3x}{1+q^3} - \frac{4q^5 \sin 5x}{1+q^5} + \cdots \\
&(23.) \quad \frac{2kK}{\pi} \cdot \cos \operatorname{coam} \frac{2Kx}{\pi} &&= \frac{4\sqrt{q} \sin x}{1+q} - \frac{4\sqrt{q^3} \sin 3x}{1+q^3} + \frac{4\sqrt{q^5} \sin 5x}{1+q^5} - \cdots
\end{aligned}
$$

The formulas (19.), (21.) by means of known expansions can also easily derived from those we gave in § 35 (6.), (7.):

$$
\begin{aligned}
\sin \operatorname{am} \frac{2Kx}{\pi} &= \frac{2\pi}{kK} \sin x \left(\frac{\sqrt{q}(1+q)}{1-2q\cos x + q^2} + \frac{\sqrt{q^3}(1+q^3)}{1-2q^3 \cos 2x + q^6} + \frac{\sqrt{q^5}(1+q^5)}{1-2q^5 \cos 2x + q^{10}} + \cdots \right) \\
\cos \operatorname{am} \frac{2Kx}{\pi} &= \frac{2\pi}{kK} \cos x \left(\frac{\sqrt{q}(1-q)}{1-2q\cos x + q^2} - \frac{\sqrt{q^3}(1-q^3)}{1-2q^3 \cos 2x + q^6} + \frac{\sqrt{q^5}(1-q^5)}{1-2q^5 \cos 2x + q^{10}} - \cdots \right)
\end{aligned}
$$

From formula (9.) § 35:

$$
\operatorname{am} \frac{2Kx}{\pi} = \pm x + 2 \arctan \frac{(1+q) \tan x}{1-q} - 2 \arctan \frac{(1+q^3) \tan x}{1-q^3} + 2 \arctan \frac{(1+q^5) \tan x}{1-q^5} - \cdots
$$

it also follows:

$$
(24.) \qquad \operatorname{am} \frac{2Kx}{\pi} = x + \frac{2q \sin 2x}{1+q^2} + \frac{2q^2 \sin 4x}{2(1+q^4)} + \frac{2q^3 \sin 6x}{3(1+q^6)} + \cdots
$$

For, the same can taking into account the ambiguous sign be represented this way:

$$\begin{array}{llll} +x+2\arctan\dfrac{(1+q)t}{1-q} & -2\arctan\dfrac{(1+q^3)t}{1-q^3} & +2\arctan\dfrac{(1+q^5)t}{1-q^5} & -\cdots \\ \quad -2x & \quad +2x & \quad -2x & +\cdots \end{array}$$

if for the sake of brevity one puts $t = \tan x$. But it is:

$$\arctan\frac{(1+q)t}{1-q} - x = \arctan\frac{(1+q)t-(1-q)t}{1-q+(1+q)tt} = \arctan\frac{2qt}{1+tt-q(1-tt)} = \arctan\frac{q\sin 2x}{1-q\cos 2x},$$

whence:

$$\operatorname{am}\frac{2Kx}{\pi} = x + 2\arctan\frac{q\sin 2x}{1-q\cos 2x} - 2\arctan\frac{q^3\sin 2x}{1-q^3\cos 2x} + 2\arctan\frac{q^5\sin 2x}{1-q^5\cos 2x} - \cdots,$$

or because it is:

$$\arctan\frac{q\sin 2x}{1-q\cos 2x} = q\sin 2x + \frac{q^2\sin 4x}{2} + \frac{q^3\sin 6x}{3} + \cdots,$$

it is:

$$\operatorname{am}\frac{2Kx}{\pi} = x + \frac{2q\sin 2x}{1+q^2} + \frac{2q^2\sin 4x}{2(1+q^4)} + \frac{2q^3\sin 6x}{3(1+q^6)} + \cdots,$$

which is formula (24.). From its differentiation is arises:

$$(25.)\qquad \frac{2K}{\pi}\cdot\Delta\operatorname{am}\frac{2Kx}{\pi} = 1 + \frac{4q\cos 2x}{1+q^2} + \frac{4q^2\cos 4x}{1+q^4} + \frac{4q^3\cos 6x}{1+q^6} + \cdots,$$

whence also having put $-q$ instead of q or $\frac{\pi}{2} - x$ instead of x:

$$(26.)\qquad \frac{2k'K}{\pi\Delta\operatorname{am}\frac{2Kx}{\pi}} = 1 - \frac{4q\cos 2x}{1+q^2} + \frac{4q^2\cos 4x}{1+q^4} - \frac{4q^3\cos 6x}{1+q^6} + \cdots$$

40.

From the propounded formulas, by putting $x = 0$ or substituting other values, the following are easily found:

$$(1.)\qquad \ln k = \ln 4\sqrt{q} - \frac{4q}{1+q} + \frac{4q^2}{2(1+q^2)} - \frac{4q^3}{3(1+q^3)} + \frac{4q^4}{4(1+q^4)} - \cdots$$

$$(2.)\qquad -\ln k' = \frac{8q}{1-q^2} + \frac{8q^3}{3(1-q^6)} + \frac{8q^5}{5(1-q^{10})} + \frac{8q^7}{7(1-q^{14})} + \cdots$$

$$(3.)\qquad \ln\frac{2K}{\pi} = \frac{4q}{1+q} + \frac{4q^3}{3(1+q^3)} + \frac{4q^5}{5(1+q^5)} + \frac{4q^7}{7(1-q^7)} + \cdots$$

And

$$(4.)\qquad \frac{2K}{\pi} = 1 + \frac{4q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \cdots$$

$$= 1 + \frac{4q}{1+q^2} + \frac{4q^2}{1+q^4} + \frac{4q^3}{1+q^6} + \cdots$$

$$(5.)\qquad \frac{2kK}{\pi} = \frac{4\sqrt{q}}{1-q} - \frac{4\sqrt{q^3}}{1-q^3} + \frac{4\sqrt{q^5}}{1-q^5} - \cdots$$

$$= \frac{4\sqrt{q}}{1+q} + \frac{4\sqrt{q^3}}{1+q^3} + \frac{4\sqrt{q^5}}{1+q^5} + \cdots$$

$$(6.)\qquad \frac{2k'K}{\pi} = 1 - \frac{4q}{1+q} - \frac{4q^3}{1+q^3} - \frac{4q^5}{1+q^5} + \cdots$$

$$= 1 - \frac{4q}{1+q^2} + \frac{4q^2}{1+q^4} + \frac{4q^3}{1+q^6} + \cdots$$

$$(7.)\qquad \frac{2\sqrt{k'}K}{\pi} = 1 - \frac{4q^2}{1+q^2} + \frac{4q^6}{1+q^6} - \frac{4q^{10}}{1+q^{10}} + \cdots$$

$$= 1 - \frac{4q^2}{1+q^4} + \frac{4q^4}{1+q^8} - \frac{4q^6}{1+q^{12}} + \cdots$$

$$(8.)\qquad \frac{4KK}{\pi\pi} = 1 + \frac{8q}{1-q} + \frac{16q^2}{1+q^2} + \frac{24q^3}{1-q^3} + \cdots$$

$$= 1 + \frac{8q}{(1-q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1-q^3)^2} + \cdots$$

$$(9.)\quad \frac{4kkKK}{\pi\pi} = \frac{16q}{1-q^2} + \frac{48q^3}{1-q^6} + \frac{80q^5}{1-q^{10}} + \cdots$$
$$= \frac{16q(1+q^2)}{(1-q^2)^2} + \frac{16q^3(1+q^6)}{(1-q^6)^2} + \frac{16q^5(1+q^{10})}{(1-q^{10})^2} + \cdots$$

$$(10.)\quad \frac{4k'k'KK}{\pi\pi} = 1 - \frac{8q}{1+q} + \frac{16q^2}{1+q^2} - \frac{24q^3}{1+q^3} + \cdots$$
$$= 1 - \frac{8q}{(1+q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1+q^3)^2} + \cdots$$

$$(11.)\quad \frac{4kk'KK}{\pi\pi} = \frac{4\sqrt{q}}{1+q} - \frac{12\sqrt{q^3}}{1+q^3} + \frac{20\sqrt{q^5}}{1+q^5} - \cdots$$
$$= \frac{4\sqrt{q}(1-q)}{(1+q)^2} - \frac{4\sqrt{q^3}(1-q^3)}{(1+q^3)^2} + \frac{4\sqrt{q^5}(1-q^5)}{(1+q^5)^2} - \cdots$$

$$(12.)\quad \frac{4k'KK}{\pi\pi} = 1 - \frac{8q^2}{1+q^2} + \frac{16q^4}{1+q^4} - \frac{24q^6}{1+q^6} + \cdots$$
$$= 1 - \frac{8q^2}{(1+q^2)^2} + \frac{8q^4}{(1+q^4)^2} + \frac{8q^6}{(1+q^6)^2} + \cdots$$

$$(13.)\quad \frac{4kKK}{\pi\pi} = \frac{4\sqrt{q}}{1-q} + \frac{12\sqrt{q^3}}{1-q^3} + \frac{20\sqrt{q^5}}{1-q^5} + \cdots$$
$$= \frac{4\sqrt{q}(1+q)}{(1-q)^2} + \frac{4\sqrt{q^3}(1+q^3)}{(1-q^3)^2} + \frac{4\sqrt{q^5}(1+q^5)}{(1-q^5)^2} + \cdots$$

We represented formulas (4.)- (13.) in two ways; but, the one representation easily follows from the other, if the single denominators are expanded into series. Further, we add, according to the theorems propounded in § 37 that from two of their total number, namely (4.) and (8.), one can derive all. For, by putting $\sqrt{q}$ instead of q, because K goes over into $(1+k)K$, by subtracting from formula (4.) then (5.) arises; secondly, by putting $-q$ instead of q, K goes over into $k'K$, whence from the formulas (4.), (8.) formulas (6.), (10.) arise; (5.) remains unchanged. By putting q^2 instead q, $k'K$ goes over into $\sqrt{k'}K$, whence from (6.), (10.) then (7.), (12.). From (8.), (10.), because $kk + k'k' = 1$, (9.) arises. By putting $\sqrt{q}$ instead of q, kK goes over into $2\sqrt{k}K$, whence (13.) arises from (9.). By putting $-q$ instead of q, kKK goes over into $ikk'KK$, whence (11.) arises from (13.). However, for the modulus itself or the complement series of such a kind do not seem to exist. Having expanded the propounded formulas into a power series in q we obtain:

$$(14.)\qquad \ln k = \ln 4\sqrt{q} - 4q + 6q^2 - \frac{16}{3}q^3 + 3q^4 - \frac{24}{5}q^5 + 8q^6 - \frac{32}{7}q^7 + \frac{3}{2}q^8 - \frac{52}{9}q^9 + \frac{36}{5}q^{10} - \cdots$$

$$(15.)\qquad -\ln k' = 8q + \frac{32}{3}q^3 + \frac{48}{5}q^5 + \frac{64}{7}q^7 + \frac{104}{9}q^9 + \frac{96}{11}q^{11} + \frac{112}{13}q^{13} + \frac{192}{15}q^{15} + \cdots$$

$$(16.)\qquad \ln\frac{2K}{\pi} = 4q - 4q^2 + \frac{16}{3}q^3 - 4q^4 + \frac{24}{5}q^5 - \frac{16}{3}q^6 + \frac{32}{7}q^7 - 4q^8 + \frac{52}{9}q^9 - \frac{24}{5}q^{10} + \cdots$$

$$(17.)\qquad \frac{2K}{\pi} = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8^{10} + 8q^{13} + 4q^{16} + 8q^{17} + 4q^{18} + \cdots$$

$$(18.)\qquad \frac{2kK}{\pi} = 4\sqrt{q} + 8\sqrt{q^5} + 4\sqrt{q^9} + 8\sqrt{q^{13}} + 8\sqrt{q^{17}} + 12\sqrt{q^{25}} + 8\sqrt{q^{29}} + 8\sqrt{q^{37}} + \cdots$$

$$(19.)\qquad \frac{2k'K}{\pi} = 1 - 4q + 4q^2 + 4q^4 - 8q^5 + 4q^8 - 4q^9 + 8q^{10} - 8q^{13} + 4q^{16} - 8q^{17} + 4q^{18} + \cdots$$

$$(20.)\qquad \frac{2\sqrt{k'}K}{\pi} = 1 - 4qq2 + 4q^4 + 4q^8 - 8q^{10} + 4q^{16} - 4q^{18} + 8q^{20} - 8q^{26} + 4q^{32} - \cdots$$

$$(21.)\qquad \frac{4KK}{\pi\pi} = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8 + \cdots$$

$$(22.)\qquad \frac{4kkKK}{\pi\pi} = 16q + 64q^3 + 96q^5 + 128q^7 + 208q^9 + 192q^{11} + 224q^{13} + 384q^{15} + \cdots$$

$$(23.)\qquad \frac{4k'k'}{\pi\pi} = 1 - 8q + 24q^2 - 32q^3 + 24q^4 - 48q^5 + 96q^6 - 64q^7 + 24q^7 - \cdots$$

$$(24.)\qquad \frac{4kk'KK}{\pi\pi} = 4\sqrt{q} - 16\sqrt{q^3} + 24\sqrt{q^5} - 32\sqrt{q^7} + 52\sqrt{q^9} - 48\sqrt{q^{11}} + 56\sqrt{q^{13}} - \cdots$$

$$(25.)\qquad \frac{4k'KK}{\pi\pi} = 1 - 8q^2 + 24q^4 - 32q^6 + 24q^8 - 48q^{10} + 96q^{12} - 64q^{14} + 24q^{16} - 104q^{18} + \cdots$$

$$(26.)\qquad \frac{4kKK}{\pi\pi} = 4\sqrt{q} + 16\sqrt{q^3} + 24\sqrt{q^5} + 32\sqrt{q^7} + 52\sqrt{q^9} + 48\sqrt{q^{11}} + 56\sqrt{q^{13}} + \cdots$$

To understand the law and the nature of these series better, we will denote them by a summation sign $\sum$ prefixed to its general term. Let us put that p is *an odd number, $\varphi(p)$ die sum of the factors of p.* Then it is:

$$(27.)\qquad \ln k = \ln 4\sqrt{q} - 4\sum\frac{\varphi(p)}{p}\left\{q^p - \frac{3q^{2p}}{p} - \frac{3}{4}q^{4p} - \frac{3}{8}q^{8p} - \frac{3}{16}q^{16p} - \cdots\right\}$$

$$(28.)\qquad -\ln k' = 8\sum\frac{\varphi(p)}{p}q^p$$

$$(29.)\qquad \ln\frac{2K}{\pi} = 4\sum\frac{\varphi(p)}{p}\left\{q^p - q^{2p} - q^{4p} - q^{8p} - q^{16p} - \cdots\right\}.$$

Further, let m be *an odd number, whose prime factors are all of the form* $4a-1$, n *an odd number, whose prime factors all have the form* $4a+1$, $\psi(n)$ *the number of factors of* n, l any arbitrary number from 0 to ∞: We obtain:

$$
\begin{array}{lll}
(30.) & \dfrac{2K}{\pi} & = 1 + 4\sum \psi(n) q^{2^l m^2 n} \\
(31.) & \dfrac{2kK}{\pi} & = \qquad 4\sum \psi(n) q^{\frac{m^2 n}{2}} \\
(32.) & \dfrac{2k'K}{\pi} & = 1 - 4\sum \psi(n) q^{m^2 n} + 4\sum \psi(n) q^{2^{l+1} m^2 n} \\
(33.) & \dfrac{2\sqrt{k'}K}{\pi} & = 1 - 4\sum \psi(n) q^{2m^2 n} + 4\sum \psi(n) q^{2^{l+2} m^2 n}
\end{array}
$$

p again denoting an odd number, $\varphi(p)$ the sum of factors of p, it is:

$$
\begin{array}{lll}
(34.) & \dfrac{4KK}{\pi\pi} & = 1 + 8\sum \varphi(p)[q^p + 3q^{2p} + 3q^{4p} + 3q^{8p} + 3q^{16p} + \cdots] \\
(35.) & \dfrac{4kkKK}{\pi\pi} & = \qquad 16\sum \varphi(p) q^p \\
(36.) & \dfrac{4k'k'KK}{\pi\pi} & = 1 + 8\sum \varphi(p)[-q^p + 3q^{2p} + 3q^{4p} + 3q^{8p} + 3q^{16p} + \cdots] \\
(37.) & \dfrac{4kk'KK}{\pi\pi} & = \qquad 4\sum (-1)^{\frac{p-1}{2}} \varphi(p) \sqrt{q^p} \\
(38.) & \dfrac{4k'KK}{\pi\pi} & = 1 + 8\sum \varphi(p)[-q^{2p} + 3q^{4p} + 3q^{8p} + 3q^{16p} + 3q^{32p} + \cdots] \\
(39.) & \dfrac{4kKK}{\pi\pi} & = \qquad 4\sum \varphi(p) \sqrt{q^p}.
\end{array}
$$

Let us demonstrate formula (27.). We found (1.):

$$\ln k = \ln 4\sqrt{q} - \frac{4q}{1+q} + \frac{4q^2}{2(1+q^2)} - \frac{4q^3}{3(1+q^4)} + \cdots,$$

which we shall put $= \ln 4\sqrt{q} + 4\sum A^{(x)} q^x$. Let x be an odd number $p = mm'$, from the general term $-\frac{q^m}{m(1+q^m)}$ it arises $\frac{-q^p}{m}$, whence it is clear that it will be $A^{(p)} = -\frac{\varphi(p)}{p}$. Now, let x be an even number $= 2^l p = 2^l mm'$: from the terms

$$\frac{-q^m}{m(1+q^m)} + \frac{q^{2m}}{2m(1+q^{2m})} + \frac{q^{4m}}{4m(1+q^{4m})} + + \frac{q^{8m}}{8m(1+q^{8m})} + \cdots + \frac{q^{2^l}}{2^l m(1+q^{2^l m})}$$

it arises:

$$\frac{q^x}{m}\left\{1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}\cdots-\frac{1}{2^{l-1}}+\frac{1}{2^l}\right\}=\frac{3q^x}{2^l m},$$

whence $A^{(x)}=\frac{3\varphi(p)}{2^l p}$ what yields the propounded formula.

Let us demonstrate formula (30.). We found (4.):

$$\frac{2K}{\pi}=1+\frac{4q}{1-q}-\frac{4q^3}{1-q^3}+\frac{4q^5}{1-q^5}-\cdots=1+4\sum A^{(x)}q^x.$$

Let $B^{(x)}$ be the number of factors of x which have the form $4m+1$, $C^{(x}$ the number of factors which have the form $4m+3$, it easily becomes clear that $A^{(x)}=B^{(x)}-C^{(x)}$. Let $x=2^l nn'$ such that n is an odd number whose prime factors all have the form $4m+1$, n' an odd number whose prime factors all have the form $4m-1$, it is easily proved, if n' is not a square number, that it will always be $B^{(x)}-C^{(x)}=0$, and if n' is square number, that it will be $B^{(x)}-C^{(x)}=\phi(n)$, whence formula (30.) follows:

Finally, let us prove (34.). We found (8.):

$$\frac{4KK}{\pi\pi}=1+\frac{8q}{1-q}+\frac{16q^2}{1+q^2}+\frac{24q^3}{1-q^3}+\frac{32q^4}{1+q^4}+\cdots=1+8\sum A^{(x)}q^x.$$

While x denotes an odd number it easily becomes clear that it will be $A^{(x)}=\varphi(x)$; but if x is an even number $=2^l p$, p denoting an odd number, if m is a factor of m, from the terms

$$8\left\{\frac{mq^m}{1-q^m}+\frac{2mq^{2m}}{1+q^{2m}}+\frac{4mq^{4m}}{1+q^{4m}}+\frac{8mq^{8m}}{1+q^{8m}}+\cdots+\frac{2^l mq^{2^l m}}{1+q^{2^l m}}\right\}$$

it arises $8mq^x\left\{1-2-4-8-\cdots-2^{l-1}+2^l\right\}=24mq^x$, whence in this case $A^{(x)}=3\varphi(p)$ what yields the propounded formula. The remaining are proven in a similar way or are deduced from these.

The expressions $\cos\operatorname{am}\frac{2Kx}{\pi}$, $\Delta\operatorname{am}\frac{2Kx}{\pi}$, $\frac{1}{\cos\operatorname{am}\frac{2Kx}{\pi}}$ expanded into a power series in x obtain as a coefficient of x^2 the expressions $-\frac{1}{2}\left(\frac{2K}{\pi}\right)^2$, $-\frac{1}{2}\left(\frac{2kK}{\pi}\right)^2$, $+\frac{1}{2}\left(\frac{2K}{\pi}\right)^2$, respectively, whence from the formulas (21.), (25.), (20.) of the preceding paragraph we see the following equations to arise:

$$(40.)\quad k\left(\frac{2K}{\pi}\right)^3 = 4\left\{\frac{\sqrt{q}}{1+q}+\frac{9\sqrt{q^3}}{1+q^3}+\frac{25\sqrt{q^5}}{1+q^5}+\frac{49\sqrt{q^7}}{1+q^7}+\cdots\right\}$$

$$= 4\left\{\frac{\sqrt{q}(1+6q+q^2)}{(1-q^3)}-\frac{\sqrt{q^3}(1+6q^3+q^6)}{(1-q^3)^3}+\frac{\sqrt{q^5}(1+6q^5+q^{10})}{(1-q^5)^3}-\cdots\right\}$$

$$(41.)\quad k'\left(\frac{2K}{\pi}\right)^3 = 1 + 4\left\{\frac{q}{1+q}-\frac{9q^3}{1+q^3}+\frac{25q^5}{1+q^5}-\frac{49q^7}{1+q^7}+\cdots\right\}$$

$$= 1 + 4\left\{\frac{q(1-6q^2+q^4)}{(1+q^2)^3}-\frac{q^2(1-6q^4+q^8)}{(1+q^4)^3}+\frac{q^3(1-6q^6+q^{12})}{(1+q^3)^3}-\cdots\right\}$$

$$(42.)\quad kk\left(\frac{2K}{\pi}\right)^3 = 16\left\{\frac{q}{1+q^2}+\frac{4q^2}{1+q^4}+\frac{9q^3}{1+q^6}+\frac{16q^4}{1+q^8}\cdots\right\}$$

$$= 16\left\{\frac{q(1+q)}{(1-q)^3}-\frac{q^3(1+q^3)}{(1-q^3)^3}+\frac{q^5(1+q^5)}{(1-q^5)^3}+\cdots\right\}$$

From these, having put $-q$ instead of q, we obtain:

$$(43.)\quad kk'k'\left(\frac{2K}{\pi}\right)^3 = 4\left\{\frac{\sqrt{q}}{1-q}-\frac{9\sqrt{q^3}}{1-q^3}+\frac{25\sqrt{q^5}}{1-q^5}-\frac{49\sqrt{q^7}}{1-q^7}+\cdots\right\}$$

$$(44.)\quad k'k'\left(\frac{2K}{\pi}\right)^3 = 1 - 4\left\{\frac{q}{1-q}-\frac{9q^3}{1-q^3}+\frac{25q^5}{1-q^5}-\frac{49q^7}{1-q^7}+\cdots\right\}$$

$$(45.)\quad k'kk\left(\frac{2K}{\pi}\right)^3 = 16\left\{\frac{q}{1+q^2}-\frac{4q^2}{1+q^4}+\frac{9q^3}{1-+q^6}-\frac{49q^4}{1+q^8}+\cdots\right\}.$$

Having added formulas (40.), (42.), we obtain $\left(\frac{2K}{\pi}\right)^3$; having subtracted (40.) from (43.), (41.) and (45.), we obtain $\left(\frac{2kK}{\pi}\right)^3$, $\left(\frac{2k'K}{\pi}\right)^3$, from which, having respectively $\sqrt{q}$, q^2 instead of q, it arises $\left(\frac{4\sqrt{k}K}{\pi}\right)^3$, $\left(\frac{4\sqrt{k'}K}{\pi}\right)^3$; from $\left(\frac{4\sqrt{k}K}{\pi}\right)^3$ having put $-q$ instead of q, one obtains $\left(\frac{4\sqrt{kk'}K}{\pi}\right)^3$.

At last, having put $k=\sin\vartheta$, let us expand $\vartheta=\arcsin k$. We saw, having put $\sqrt{q}$ instead of q, that k' goes over into $\frac{1-k}{1+k}$; let us again put $-q$ instead of q, k then goes over into $\frac{ik}{k'}$, or into $i\tan\vartheta$, such that having put $i\sqrt{q}$ instead of q the expression $-\frac{\ln k'}{2i}$ is changed to:

$$-\frac{1}{2i}\ln\left(\frac{1-i\tan\vartheta}{1+i\tan\vartheta}\right)=\vartheta.$$

Hence, from formula (2.):

$$-\ln k'=\frac{8q}{1-q^2}+\frac{8q^3}{3(1-q^6)}+\frac{8q^5}{5(1-q^{10})}+\frac{8q^7}{7(1-q^{14})}+\cdots$$

we find:

$$(46.)\qquad \vartheta=\arcsin k=\frac{4\sqrt{q}}{1+q}-\frac{4\sqrt{q^3}}{3(1+q^3)}+\frac{4\sqrt{q^5}}{5(1+q^5)}-\frac{4\sqrt{q^7}}{7(1+q^7)}+\cdots,$$

which is easily transformed into this one:

$$(47.)\qquad \frac{\vartheta}{4}=\arctan\sqrt{q}-\arctan\sqrt{q^3}+\arctan\sqrt{q^5}-\arctan\sqrt{q^7}+\cdots,$$

which is to counted among the most elegant formulas.

41.

Let us multiply the equation exhibited above:

$$\frac{2kK}{\pi}\sin\operatorname{am}\frac{2Kx}{\pi}=\frac{4\sqrt{q}\sin x}{1-q}+\frac{4\sqrt{q^3}\sin 3x}{1-q^3}+\frac{4\sqrt{q^5}\sin 5x}{1-q^5}+\cdots$$

by itself. Having substituted for $2\sin mx\sin nx$ everywhere the expression

$$\cos(m-n)x-\cos(m+n)x$$

it takes the following form:

$$\left(\frac{2kK}{\pi}\right)^2\sin^2\operatorname{am}\frac{2Kx}{\pi}=A+A'\cos 2x+A''\cos 4x+A'''\cos 6x+\cdots$$

It is found:

$$A=\frac{8q}{(1-q)^2}+\frac{8q^3}{(1-q^3)^2}+\frac{8q^5}{(1-q^5)^2}+\cdots$$

Further, it is:

$$A^{(n)} = 16B^{(n)} - 8C^{(n)} = 8[2B^{(n)} - C^{(n)}],$$

if it is put:

$$B^{(n)} = \frac{q^{n+1}}{(1-q)(1-q^{2n+1})} + \frac{q^{n+3}}{(1-q^3)(1-q^{2n+3})} + \frac{q^{n+5}}{(1-q^5)(1-q^{2n+5})} + \text{etc. to infinity}$$

$$C^{(n)} = \frac{q^n}{(1-q)(1-q^{2n-1})} + \frac{q^n}{(1-q^3)(1-q^{2n-3})} + \frac{q^n}{(1-q^5)(1-q^{2n-5})} + \cdots + \frac{q^n}{(1-q^{2n-1})(1-q)}.$$

Now, because it is:

$$\frac{q^{m+n}}{(1-q^m)(1-q^{2m+n})} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^m}{1-q^m} - \frac{q^{2n+m}}{1-q^{2n+m}}\right\}$$

it becomes:

$$B^{(n)} = \left\{\begin{array}{l} \dfrac{q^n}{1-q^{2n}}\left\{\dfrac{q}{1-q} + \dfrac{q^3}{1-q^3} + \dfrac{q^5}{1-q^5} + \cdots\right\} \\ -\dfrac{q^n}{1-q^{2n}}\left\{\dfrac{q^{2n+1}}{1-q^{2n+1}} + \dfrac{q^{2n+3}}{1-q^{2n+3}} + \dfrac{q^{2n+5}}{1-q^{2n+5}} + \cdots\right\} \end{array}\right\}$$

or having cleared the terms which cancel:

$$B^{(n)} = \frac{q^n}{1-q^{2n}}\left\{\frac{q}{1-q} + \frac{q^3}{1-q^3} + \cdots + \frac{q^{2n-1}}{1-q^{2n-1}}\right\}.$$

Further, it is:

$$\frac{q^n}{(1-q^m)(1-q^{2n-m})} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^m}{1-q^m} + \frac{q^{2n-m}}{1-q^{2n-m}} + 1\right\},$$

whence:

$$C^{(n)} = \frac{nq^n}{1-q^{2n}} + \frac{2q^n}{1-q^{2n}}\left\{\frac{q}{1-q} + \frac{q^3}{1-q^3} + \cdots + \frac{q^{2n-1}}{1-q^{2n-1}}\right\}.$$

Hence, it finally arises:

$$A^{(n)} = 8[2B^{(n)} - C^{(n)}] = \frac{-8nq^n}{1-q^{2n}},$$

whence now:

$$(1.)\qquad \left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi} = A - 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\}.$$

In similar manner it is also found from (1.):

$$(2.)\qquad \left(\frac{2kK}{\pi}\right)^2 \cos^2 \operatorname{am} \frac{2Kx}{\pi} = B + 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\},$$

if:

$$\begin{aligned} A &= 8\left\{\frac{q}{(1-q)^2} + \frac{q^3}{(1-q^3)} + \frac{q^5}{(1-q^5)} + \cdots\right\} \\ B &= 8\left\{\frac{q}{(1+q)^2} + \frac{q^3}{(1+q^3)} + \frac{q^5}{(1+q^5)} + \cdots\right\}. \end{aligned}$$

From a known theorem of integral, if

$$\varphi(x) = A + A'\cos 2x + A''\cos 4x + A'''\cos 6x + \cdots,$$

the constant or first term becomes:

$$A = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi(x)dx$$

whence we obtain in this case:

$$\begin{aligned} A &= \frac{2}{\pi}\left(\frac{2kK}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \sin^2 \operatorname{am} \frac{2Kx}{\pi} dx \\ B &= \frac{2}{\pi}\left(\frac{2kK}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \cos^2 \operatorname{am} \frac{2Kx}{\pi} dx. \end{aligned}$$

Following Legendre, let is put:

$$E^{\mathrm{I}} = \int_0^{\frac{\pi}{2}} d\varphi \Delta(\varphi) = \frac{2K}{\pi} \int_0^{\frac{\pi}{2}} dx \Delta^2 \operatorname{am} \frac{2Kx}{\pi},$$

it will be:

$$\begin{aligned} A &= \frac{2K}{\pi} \cdot \frac{2K}{\pi} - \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} \\ B &= \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} - \left(\frac{2k'K}{\pi} \right)^2 . \end{aligned}$$

Hence, also, because having changed q into $-q$ A goes over into $-q$, K into $k'K$, it follows that at the same time E^{I} goes over into $\frac{E^{\mathrm{I}}}{k'}$.

Finally, let us add that from formula (1.) it follows:

$$\begin{aligned} (3.) \quad kk\left(\frac{2K}{\pi}\right)^4 &= 16\left\{\frac{q}{1-q^2} + \frac{2^3q^2}{1-q^4} + \frac{3^3q^3}{1-q^6} + \frac{4^3q^4}{1-q^8} + \cdots\right\} \\ &= 16\left\{\frac{q(1+4q+q^2)}{(1-q)^4} + \frac{q^3(1+4q^3+q^6)}{(1-q^3)^4} + \frac{q^5(1+4q^5+q^{10})}{(1-q^5)} + \cdots\right\}, \end{aligned}$$

whence also having changed q to $-q$:

$$\begin{aligned} (4.) \quad k^2k\left(\frac{2K}{\pi}\right)^4 &= 16\left\{\frac{q}{1-q^2} - \frac{2^3q^2}{1-q^4} + \frac{3^3q^3}{1-q^6} - \frac{4^3q^4}{1-q^8} + \cdots\right\} \\ &= 16\left\{\frac{q(1-4q+q^2)}{(1+q)^4} + \frac{q^3(1+4q^3+q^6)}{(1+q^3)^4} + \frac{q^5(1-4q^5+q^{10})}{(1+q^5)} + \cdots\right\}. \end{aligned}$$

Having subtracted formula (4.) from (3.) it arises:

$$\begin{aligned} (5.) \quad \left(\frac{2kK}{\pi}\right)^4 &= 256\left\{\frac{q^2}{1-q^4} + \frac{2^3q^4}{1-q^8} + \frac{3^3q^6}{1-q^{12}} + \frac{4^3q^8}{1-q^{16}} + \cdots\right\} \\ &= 256\left\{\frac{q^2(1+4q^2+q^4)}{(1-q^2)^4} + \frac{q^6(1+4q^6+q^{12})}{(1-q^6)^4} + \frac{q^{10}(1+4q^{10}+q^{20})}{(1-q^{10})} + \cdots\right\}, \end{aligned}$$

which one obtains also from (3.) having changed q into q^2.

42.

By a similar method as formula (1.) of the preceding § was found we could have investigated how to expand the expression

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2\operatorname{am}\frac{2Kx}{\pi}}$$

into a series, if formula (18.) § 39 is multiplied by itself. This is however easier done from (1.) of § 41 itself having considered the following.

For, having differentiated formula:

$$\frac{d\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx} = \frac{2K}{\pi}\cdot\frac{\sqrt{1-(1+kk)\sin^2\operatorname{am}\frac{2Kx}{\pi}+kk\sin^4\operatorname{am}\frac{2Kx}{\pi}}}{\sin\operatorname{am}\frac{2Kx}{\pi}}$$

one more time and having done the reductions, we obtain:

$$(1.)\qquad \frac{d^2\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx^2} = \left(\frac{2K}{\pi}\right)^2\left\{kk\sin^2\operatorname{am}\frac{2Kx}{\pi} - \frac{1}{\sin^2\operatorname{am}\frac{2Kx}{\pi}}\right\}.$$

But we already found in § 39 (6.):

$$\ln\sin\operatorname{am}\frac{2Kx}{\pi} = \ln\left(\frac{2\sqrt[4]{q}}{\sqrt{k}}\right) + \ln\sin x + 2\left\{\frac{q\cos 2x}{1+q} + \frac{q^2\cos 4x}{2(1+q^2)} + \frac{q^3\cos 6x}{3(1+q^3)} + \cdots\right\},$$

whence:

$$\frac{d^2\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx^2} = -\frac{1}{\sin^2 x} - 8\left\{\frac{q\cos 2x}{1+q} + \frac{2q^2\cos 4x}{1+q^2} + \frac{3q^3\cos 6x}{1+q^3} + \cdots\right\}.$$

Further, it is § 41 (1.):

$$\left(\frac{2kK}{\pi}\right)^2\sin^2\operatorname{am}\frac{2Kx}{\pi} = \frac{2K}{\pi}\cdot\frac{2K}{\pi} - \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} - 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\},$$

whence, because it is from formula (1.):

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2\operatorname{am}\frac{2Kx}{\pi}} = \left(\frac{2kK}{\pi}\right)^2 \sin^2\operatorname{am}\frac{2Kx}{\pi} - \frac{d^2 \ln \sin\operatorname{am}\frac{2Kx}{\pi}}{dx^2},$$

it arises what we are looking for:

$$(2.)\quad \frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2\operatorname{am}\frac{2Kx}{\pi}} = \frac{2K}{\pi}\cdot\frac{2K}{\pi} - \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} + \frac{1}{\sin^2 x} - 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\}$$

Having changed at the same time q to $-q$ as x to $\frac{\pi}{2}-x$, whence K goes over into $k'K$, E^{I} into $\frac{E^{\mathrm{I}}}{k'}$ (§ 41), $\sin\operatorname{am}\frac{2Kx}{\pi}$ into $\cos\operatorname{am}\frac{2Kx}{\pi}$, it arises from (2.):

$$(3.)\quad \frac{\left(\frac{2k'K}{\pi}\right)^2}{\cos^2\operatorname{am}\frac{2Kx}{\pi}} = \left(\frac{2k'K}{\pi}\right)^2 - \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} + \frac{1}{\cos^2 x} + 8\left\{\frac{q^2\cos 2x}{1-q^2} - \frac{2q^4\cos 4x}{1-q^4} + \frac{3q^6\cos 6x}{1-q^6} - \cdots\right\}$$

To these I add these easily following from § 41. (1.):

$$(4.)\quad \left(\frac{2K}{\pi}\right)^2 \Delta^2\operatorname{am}\frac{2Kx}{\pi} = \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} + 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\}$$

$$(5.)\quad \left(\frac{2k'K}{\pi}\right)^2 \frac{1}{\Delta^2\operatorname{am}\frac{2Kx}{\pi}} = \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} - 8\left\{\frac{q\cos 2x}{1-q^2} - \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} - \cdots\right\}$$

of which (5.) follows from (4.) having changed x into $\frac{\pi}{2}-x$ or q into $-q$.

Having put $y = \sin\operatorname{am}\frac{2Kx}{\pi}$, $\sqrt{(1-y^2)(1-k^2y^2)} = R$ it is:

$$\begin{aligned}
\frac{dy}{dx} &= \left(\frac{2K}{\pi}\right)\cdot R \\
\frac{d^2y}{dx^2} &= -\left(\frac{2K}{\pi}\right)^2 y(1+k^2-2k^2y^2) \\
\frac{d^3y}{dx^3} &= -\left(\frac{2K}{\pi}\right)^3 (1+k^2-6k^2y^2)R \\
\frac{d^4y}{dx^4} &= \left(\frac{2K}{\pi}\right)^4 y(1+14k^2+k^4-20k^3(1+k^2)y^2+24k^4y^4) \\
\frac{d^5y}{dx^5} &= \left(\frac{2K}{\pi}\right)^5 (1+14k^2+k^4-60k^2(1+k^2)y^2+120k^4y^4)R \\
& \text{etc.} \qquad \text{etc.,}
\end{aligned}$$

whence:

$$y = \sin\operatorname{am}\frac{2Kx}{\pi} = \frac{2Kx}{\pi} - \frac{1+k^2}{2\cdot 3}\left(\frac{2Kx}{\pi}\right)^3 + \frac{1+14k^2+k^4}{2\cdot 3\cdot 4\cdot 5}\left(\frac{2Kx}{\pi}\right)^5 - \cdots$$

and hence:

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2\operatorname{am}\frac{2Kx}{\pi}} = \frac{1}{x^2} + \frac{1+k^2}{3}\left(\frac{2K}{\pi}\right)^2 + \frac{1-k^2+k^4}{15}\left(\frac{2K}{\pi}\right)^2 x^2 + \cdots,$$

after having compared which formula to (2.) it is found:

$$\frac{1+k^2}{3}\left(\frac{2K}{\pi}\right)^2 = \frac{1}{3} + \left(\frac{2K}{\pi}\right)^2 - \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} - 8\left\{\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \cdots\right\},$$

or

$$(6.) \qquad \frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \frac{4q^8}{1-q^8} + \cdots = \frac{1+\left(\frac{2K}{\pi}\right)^2(2-k^2) - 3\frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi}}{2\cdot 3\cdot 4}.$$

Further, it is:

$$\frac{1-k^2+k^4}{15}\left(\frac{2K}{\pi}\right)^4 = \frac{1}{15} + 16\left\{\frac{q^2}{1-q^2} + \frac{2^3q^4}{1-q^4} + \frac{3^3q^6}{1-q^6} + \frac{4^3q^8}{1-q^8} + \cdots\right\}$$

or because it is $15 = 2\cdot 2^3 - 1$:

$$\begin{aligned}(1-k^2+k^4)\left(\frac{2K}{\pi}\right)^4 &= 1 + 2\cdot 16\left\{\frac{2^3q^2}{1-q^2} + \frac{4^3q^4}{1-q^4} + \frac{6^3q^6}{1-q^6} + \frac{8^3q^8}{1-q^8} + \cdots\right\}\\ &\quad - 1\cdot 16\left\{\frac{q^2}{1-q^2} + \frac{2^3q^4}{1-q^4} + \frac{3^3q^6}{1-q^6} + \frac{4^3q^8}{1-q^8} + \cdots\right\}.\end{aligned}$$

From this formula subtract the following § 41 (3.):

$$k^2\left(\frac{2K}{\pi}\right)^4 = 16\left\{\frac{q}{1-q^2} + \frac{2^3q^2}{1-q^4} + \frac{3^3q^3}{1-q^6} + \frac{4^3q^4}{1-q^8} + \cdots\right\}$$

the residue is:

$$(7.)\qquad \left(\frac{2k'K}{\pi}\right)^4 = 1 - 16\left\{\frac{q}{1-q} - \frac{2^3q^2}{1-q^2} + \frac{3^3q^3}{1-q^3} - \frac{4^3q^4}{1-q^4} + \cdots\right\},$$

whence also having changed q to $-q$:

$$(8.)\qquad \left(\frac{2K}{\pi}\right)^4 = 1 + 16\left\{\frac{q}{1+q} + \frac{2^3q^2}{1-q^2} + \frac{3^3q^3}{1+q^3} + \frac{4^3q^4}{1-q^4} + \cdots\right\},$$

which formula were more difficult to find. If one combines them with those we found above one now has the first four powers of $\frac{2K}{\pi}$, $\frac{2kK}{\pi}$ expanded into a rather beautiful series.

2.2 General Formulas for the Expansion of the Functions $\sin^n \operatorname{am} \frac{2Kx}{\pi}$, $\frac{1}{\sin^n \operatorname{am} \frac{2Kx}{\pi}}$ into a Series of Sines and Cosines of Multiples of x

43.

Having found the expansions of the functions

$$\sin\operatorname{am}\frac{2Kx}{\pi},\quad \sin^2\operatorname{am}\frac{2Kx}{\pi},\quad \frac{1}{\sin\operatorname{am}\frac{2Kx}{\pi}},\quad \frac{1}{\sin^2\operatorname{am}\frac{2Kx}{\pi}},$$

this automatically raises the question on the expansion of the powers of

$$\sin\operatorname{am}\frac{2Kx}{\pi},\quad \frac{1}{\sin\operatorname{am}\frac{2Kx}{\pi}}.$$

There is an easy way via analytic geometry following which having found the expansion of $\sin x$, $\cos x$ you can proceed to the expansion of the expressions $\cos^n x$, $\sin^n x$; this is certainly achieved by means of the known formulas according which $\sin^n x$ and $\cos^n x$ are expression as linear combinations of sines and cosines of multiples of x. But in the theory of elliptic function such an auxiliary tool does not exist; one will have to use another way we will explain in the following.

Having differentiated the formula obvious from the elements:

$$\frac{d\sin^n\operatorname{am}u}{du} = n\sin^{n-1}\operatorname{am}u\sqrt{1-(1+k^2)\sin^2\operatorname{am}u+k^2\sin^4\operatorname{am}u}$$

one more time it arises:

$$(1.)\qquad \frac{d^2\sin^n\operatorname{am}u}{du^2} = n(n-1)\sin^{n-2}\operatorname{am}u - n^2(1+k^2)\sin^n\operatorname{am}u + n(n+1)k^2\sin^{n+2}\operatorname{am}u.$$

Having successively put $n = 1, 3, 5, 7\cdots$, $n = 2, 4, 6, 8, \cdots$, from this then form two series of equations:

I.

$$\begin{aligned}
\frac{d^2\sin\operatorname{am}u}{du^2} &= \phantom{6\sin\operatorname{am}u} - 1(1+k^2)\sin\operatorname{am}u + 2k^2\sin^3\operatorname{am}u\\
\frac{d^2\sin^3\operatorname{am}u}{du^2} &= 6\sin\operatorname{am}u - 9(1+k^2)\sin^3\operatorname{am}u + 12k^2\sin^5\operatorname{am}u\\
\frac{d^2\sin^5\operatorname{am}u}{du^2} &= 20\sin^3\operatorname{am}u - 25(1+k^2)\sin^5\operatorname{am}u + 30k^2\sin^7\operatorname{am}u\\
\frac{d^2\sin^7\operatorname{am}u}{du^2} &= 42\sin^5\operatorname{am}u - 49(1+k^2)\sin^7\operatorname{am}u + 56k^2\sin^9\operatorname{am}u
\end{aligned}$$

etc. etc.

II.

$$\frac{d^2 \sin^2 \operatorname{am} u}{du^2} = 2 \qquad\qquad - 4(1+k^2) \ \sin \operatorname{am} u \ + 6k^2 \sin^4 \operatorname{am} u$$
$$\frac{d^2 \sin^4 \operatorname{am} u}{du^2} = 12 \sin^2 \operatorname{am} u - 16(1+k^2) \sin^4 \operatorname{am} u + 20k^2 \sin^6 \operatorname{am} u$$
$$\frac{d^2 \sin^6 \operatorname{am} u}{du^2} = 30 \sin^4 \operatorname{am} u - 36(1+k^2) \sin^6 \operatorname{am} u + 42k^2 \sin^8 \operatorname{am} u$$
$$\frac{d^2 \sin^8 \operatorname{am} u}{du^2} = 56 \sin^6 \operatorname{am} u - 64(1+k^2) \sin^8 \operatorname{am} u + 72k^2 \sin^{10} \operatorname{am} u$$

etc. etc.

From equations I., II. you successively find having put $\Pi n = 1 \cdot 2 \cdot 3 \cdots n$:

I. a.

$$\Pi 2 \cdot k^2 \sin^3 \operatorname{am} u = \frac{d^2 \sin \operatorname{am} u}{du^2} + (1+k^2) \sin \operatorname{am} u$$
$$\Pi 4 \cdot k^4 \sin^5 \operatorname{am} u = \frac{d^4 \sin \operatorname{am} u}{du^4} + 10(1+k^2) \frac{\sin \operatorname{am} u}{du^2} + 3(3+2k^2+3k^4) \sin \operatorname{am} u$$
$$\Pi 6 \cdot k^6 \sin^7 \operatorname{am} u = \frac{d^6 \sin \operatorname{am} u}{du^6} + 35(1+k^2) \frac{d^4 \sin \operatorname{am} u}{du^4} + 7(37+38k^2+37k^4) \frac{d^2 \sin \operatorname{am} u}{du^2}$$
$$+ 45(5+3k^2+3k^4+5k^6) \sin \operatorname{am} u$$
$$\Pi 8 \cdot k^8 \sin^9 \operatorname{am} u = \frac{d^8 \sin \operatorname{am} u}{du^8} + 84(1+k^2) \frac{d^6 \sin \operatorname{am} u}{du^6} + 42(47+58k^2+47k^4) \frac{d^4 \sin \operatorname{am} u}{du^4}$$
$$+ 4(3229+3315k^2+3315k^4+3229k^6) \frac{d^2 \sin \operatorname{am} u}{du^2}$$
$$+ 315(35+20k^2+18k^4+20k^6+35k^8) \sin \operatorname{am} u$$

etc. etc.

II. a

$$\Pi 3 \cdot k^2 \sin^4 \operatorname{am} u = \frac{d^2 \sin^2 \operatorname{am} u}{du^2} + 4(1+k^2)\sin^2 \operatorname{am} u - 2$$

$$\Pi 5 \cdot k^4 \sin^6 \operatorname{am} u = \frac{d^4 \sin^2 \operatorname{am} u}{du^4} + 20(1+k^2)\frac{d^2 \sin^2 \operatorname{am} u}{du^2} + 8(8+7k^2+8k^4)\sin^2 \operatorname{am} u - 32(1+k^2)$$

$$\Pi 7 \cdot k^6 \sin^8 \operatorname{am} u = \frac{d^6 \sin^2 \operatorname{am} u}{du^6} + 56(1+k^2)\frac{d^4 \sin^2 \operatorname{am} u}{du^4} + 112(7+8k^2+7k^4)\frac{d^2 \sin^2 \operatorname{am} u}{du^2}$$
$$+ 128(18+15k^2+15k^4+18k^6)\sin^2 \operatorname{am} u - 48(24+23k^2+24k^4)$$

etc. etc.

So we see that we can put in general:

$$(2.) \qquad \Pi(2n) \cdot k^{2n} \sin^{2n+1} \operatorname{am} u$$
$$= \frac{d^{2n} \sin \operatorname{am} u}{du^{2n}} + A_n^{(1)} \frac{d^{2n-2} \sin \operatorname{am} u}{du^{2n-2}} + A_n^{(2)} \frac{d^{2n-4} \sin \operatorname{am} u}{du^{2n-4}} + \cdots + A_n^{(n)} \sin \operatorname{am} u$$

$$(3.) \qquad \Pi(2n-1) \cdot k^{2n-2} \sin^{2n} \operatorname{am} u$$
$$= \frac{d^{2n-2} \sin^2 \operatorname{am} u}{du^{2n-2}} + B_n^{(1)} \frac{d^{2n-4} \sin^2 \operatorname{am} u}{du^{2n-4}} + B_n^{(2)} \frac{d^{2n-6} \sin^2 \operatorname{am} u}{du^{2n-6}} + \cdots + B_n^{(n-1)} \sin^2 \operatorname{am} u + B_n^{(n)},$$

where $A_n^{(m)}$, $B_n^{(m)}$ denote polynomial functions of k^2 of $m-$th order, except for $B_n^{(n)}$ which is of $(n-2)-$th order. Further, from the general formula from which we started:

$$\frac{d^2 \sin^n \operatorname{am} u}{du^2} = n(n-1)\sin^{n-2} \operatorname{am} u - n^2(1+k^2)\sin^n \operatorname{am} u + n(n+1)k^2 \sin^{n+2} \operatorname{am} u$$

it is clear that it will be:

$$(4.) \qquad A_n^{(m)} = A_{n-1}^{(m)} + (2n-1)^2(1+k^2)A_{n-1}^{(m-1)} - (2n-2)^2(2n-1)(2n-3)k^2 A_{n-2}^{(m-2)}$$
$$(5.) \qquad B_n^{(m)} = B_{n-1}^{(m)} + (2n-2)^2(1+k^2)B_{n-1}^{(m-1)} - (2n-3)^2(2n-2)(2n-4)k^2 B_{n-2}^{(m-2)},$$

in which formulas, if $m > n$, one has to put $A_n^{(m)} = 0$, $B_n^{(m)} = 0$.

Having changed u into $u + iK'$, since $\sin \operatorname{am} u$ goes over into $\frac{1}{k \sin \operatorname{am} u}$, in the propounded formulas one will be able to put $\frac{1}{\sin\operatorname{am} u}$ instead of $\sin\operatorname{am} u$, whence the following formulas arise:

$$\begin{aligned}
\frac{\Pi 2}{\sin^3\operatorname{am} u} &= \frac{d^2}{du^2}\frac{1}{\sin\operatorname{am} u} + 1(1+k^2)\frac{1}{\sin\operatorname{am} u} \\
\frac{\Pi 3}{\sin^4\operatorname{am} u} &= \frac{d^2}{du^2}\frac{1}{\sin^2\operatorname{am} u} + 4(1+k^2)\frac{1}{\sin^2\operatorname{am} u} - 2k^2 \\
\frac{\Pi 4}{\sin^5\operatorname{am} u} &= \frac{d^4}{du^4}\frac{1}{\sin\operatorname{am} u} + 10(1+k^2)\frac{d^2}{du^2}\frac{1}{\sin\operatorname{am} u} + \frac{3(3+2k^2+3k^4)}{\sin\operatorname{am} u} \\
\frac{\Pi 5}{\sin^6\operatorname{am} u} &= \frac{d^4}{du^4}\frac{1}{\sin^2\operatorname{am} u} + 20(1+k^2)\frac{d^2}{du^2}\frac{1}{\sin^2\operatorname{am} u} + \frac{8(8+7k^2+8k^4)}{\sin^2\operatorname{am} u} - 32k^2(1+k^2) \\
&\qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.,}
\end{aligned}$$

and in general

$$(6.) \qquad \frac{\Pi 2n}{\sin^{2n+1}\operatorname{am} u}$$

$$= \frac{d^{2n}}{du^{2n}}\frac{1}{\sin\operatorname{am} u} + A_n^{(1)}\frac{d^{2n-2}}{du^{2n-2}}\frac{1}{\sin\operatorname{am} u} + A_n^{(2)}\frac{d^{2n-4}}{du^{2n-4}}\frac{1}{\sin\operatorname{am} u} + \cdots + A_n^{(n)}\frac{1}{\sin\operatorname{am} u}$$

$$(7.) \qquad \frac{\Pi(2n-1)}{\sin^{2n}\operatorname{am} u}$$

$$= \frac{d^{2n-2}}{du^{2n-2}}\frac{1}{\sin^2\operatorname{am} u} + B_n^{(1)}\frac{d^{2n-4}}{du^{2n-4}}\frac{1}{\sin^2\operatorname{am} u} + B_n^{(2)}\frac{d^{2n-6}}{du^{2n-6}}\frac{1}{\sin^2\operatorname{am} u} + \cdots + B_n^{(n)}\frac{1}{\sin^2\operatorname{am} u}$$

44.

Because it was found in the preceding, if one puts $u = \frac{2Kx}{\pi}$, that the expressions

$$\sin^n\operatorname{am}\frac{2Kx}{\pi}, \quad \frac{1}{\sin^n\operatorname{am}\frac{2Kx}{\pi}}$$

can be expressed as a linear combination of these:

$$\sin\operatorname{am}\frac{2Kx}{\pi}, \quad \sin^2\operatorname{am}\frac{2Kx}{\pi}, \quad \frac{1}{\sin\operatorname{am}\frac{2Kx}{\pi}}, \quad \frac{1}{\sin^2\operatorname{am}\frac{2Kx}{\pi}}$$

and its differentials, taken with respect to the argument u or x, from their expansion into a series of sines and cosines of multiples of the argument x, the corresponding expansions immediately follow.

This way we obtain:

I.

from the formula:

$$\frac{2kK}{\pi}\sin\operatorname{am}\frac{2Kx}{\pi} = 4\left\{\frac{\sqrt{q}\sin x}{1-q} + \frac{\sqrt{q^3}\sin 3x}{1-q^3} + \frac{\sqrt{q^5}\sin 5x}{1-q^5} + \cdots\right\}$$

the following:

$$2\left(\frac{2kK}{\pi}\right)^3 \sin^3\operatorname{am}\frac{2Kx}{\pi}$$

$$\begin{aligned}
&2\left(\frac{2kK}{\pi}\right)^3 \sin^3\operatorname{am}\frac{2Kx}{\pi}\\
&= 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 1^2\right\}\frac{\sqrt{q}\sin x}{1-q}\\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 3^2\right\}\frac{\sqrt{q^3}\sin 3x}{1-q^3}\\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 5^2\right\}\frac{\sqrt{q^5}\sin 5x}{1-q^5}\\
&+ \cdots
\end{aligned}$$

$$\begin{aligned}
&2\cdot 3\cdot 4\left(\frac{2kK}{\pi}\right)^5 \sin^5\operatorname{am}\frac{2Kx}{\pi}\\
&= 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 1^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 1^4\right\}\frac{\sqrt{q}\sin x}{1-q}\\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 3^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 3^4\right\}\frac{\sqrt{q^3}\sin 3x}{1-q^3}\\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 5^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 5^4\right\}\frac{\sqrt{q^5}\sin 5x}{1-q^5}\\
&+ \cdots
\end{aligned}$$

etc. etc.

II.

from the formula:

$$\left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi} = \frac{2K}{\pi}\cdot\frac{2K}{\pi} - \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} - 4\left\{\frac{2q\cos 2x}{1-q^2} + \frac{4q^2\cos 4x}{1-q^4} + \frac{6q^3\cos 6x}{1-q^6} + \cdots\right\}$$

the following:

$$\begin{aligned}
&2\cdot 3\left(\frac{2kK}{\pi}\right)^4 \sin^4 \operatorname{am} \frac{2Kx}{\pi}\\
&= 4(1+k^2)\left(\frac{2K}{\pi}\right)^3\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) - 2k^2\left(\frac{2K}{\pi}\right)^4\\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 2^3\right\} - \frac{q\cos 2x}{1-q^2}\\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 4^3\right\} - \frac{q^2\cos 4x}{1-q^4}\\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 6^3\right\} - \frac{q^3\cos 6x}{1-q^6}\\
&- \cdots
\end{aligned}$$

$$\begin{aligned}
&2\cdot 3\cdot 4\cdot 5\left(\frac{2kK}{\pi}\right)^6 \sin^6 \operatorname{am} \frac{2Kx}{\pi}\\
&= 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^5\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) - 32k^2(1+k^2)\left(\frac{2K}{\pi}\right)^3\\
&- 4\left\{2\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 2^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 2^5\right\}\frac{q\cos 2x}{1-q^2}\\
&- 4\left\{4\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 4^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 4^5\right\}\frac{q^2\cos 4x}{1-q^4}\\
&- 4\left\{6\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 6^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 6^5\right\}\frac{q^3\cos 6x}{1-q^6}\\
&- \cdots
\end{aligned}$$

etc. etc.

III.

from the formula:

$$\frac{\frac{2K}{\pi}}{\sin\operatorname{am}\frac{2Kx}{\pi}} = \frac{1}{\sin x} + \frac{4q\sin x}{1-q} + \frac{4q^3\sin 3x}{1-q^3} + \frac{4q^5\sin 5x}{1-q^5} + \text{etc.}$$

the following:

$$\begin{aligned}
&\frac{2\left(\frac{2K}{\pi}\right)^3}{\sin^3\operatorname{am}\frac{2Kx}{\pi}}\\
&= (1+k^2)\left(\frac{2K}{\pi}\right)^2\frac{1}{\sin x} + \frac{d^2}{dx^2}\\
&+4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 1^2\right\}\frac{q\sin x}{1-q}\\
&+4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 3^2\right\}\frac{q^3\sin 3x}{1-q^3}\\
&+4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 5^2\right\}\frac{q^5\sin 5x}{1-q^5}\\
&+\cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{2\cdot 3\cdot 4\left(\frac{2K}{\pi}\right)^5}{\sin^5\operatorname{am}\frac{2Kx}{\pi}}\\
&= \frac{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4}{\sin x} + 10(1+k^2)\left(\frac{2K}{\pi}\right)^2\frac{d^2}{dx^2}\frac{1}{\sin x} + \frac{d^4}{dx^4}\frac{1}{\sin x}\\
&+4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 1^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 1^4\right\}\frac{q\sin x}{1-q}\\
&+4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 3^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 3^4\right\}\frac{q^3\sin 3x}{1-q^3}\\
&+4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 5^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 5^4\right\}\frac{q^5\sin 5x}{1-q^5}\\
&+\cdots
\end{aligned}$$

etc. etc.

IV.

from the formula:

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2 \operatorname{am} \frac{2Kx}{\pi}}$$
$$= \frac{2K}{\pi}\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) + \frac{1}{\sin^2 x} - 4\left\{\frac{2q^2\cos 2x}{1-q^2} + \frac{4q^4\cos 4x}{1-q^4} + \frac{6q^6\cos 6x}{1-q^6} + \cdots\right\}$$

the following:

$$\begin{aligned}
&\frac{2\cdot 3\left(\frac{2K}{\pi}\right)^4}{\sin^4 \operatorname{am} \frac{2Kx}{\pi}} \\
&= 4(1+k^2)\left(\frac{2K}{\pi}\right)^3\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) - 2k^2\left(\frac{2K}{\pi}\right)^4 \\
&+ \frac{4(1+k^2)\left(\frac{2K}{\pi}\right)^2}{\sin^2 x} + \frac{d^2}{dx^2}\frac{1}{\sin^2 x} \\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 2^3\right\}\frac{q^2\cos 2x}{1-q^2} \\
&- 4\left\{4\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 4^3\right\}\frac{q^4\cos 4x}{1-q^4} \\
&- 4\left\{6\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 6^3\right\}\frac{q^6\cos 6x}{1-q^6} \\
&- \cdots
\end{aligned}$$

$$
\begin{aligned}
&\frac{2\cdot 3\cdot 4\cdot 5\left(\frac{2K}{\pi}\right)^6}{\sin^6 \operatorname{am}\frac{2Kx}{\pi}}\\
&= 8(8+7k^2+8k^4)\left(\frac{2k}{\pi}\right)^5\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right)-32k^2(1+k^2)\left(\frac{2K}{\pi}\right)^6\\
&+\frac{8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4}{\sin^2 x}+20(1+k^2)\left(\frac{2K}{\pi}\right)^2\frac{d^2}{dx^2}\frac{1}{\sin^2 x}+\frac{d^4}{dx^4}\frac{1}{\sin^2 x}\\
&-4\left\{2\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4-2^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2+2^5\right\}\frac{q^2\cos 2x}{1-q^2}\\
&-4\left\{4\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4-4^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2+4^5\right\}\frac{q^4\cos 4x}{1-q^4}\\
&-4\left\{6\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4-6^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2+6^5\right\}\frac{q^6\cos 6x}{1-q^6}\\
&-\cdots
\end{aligned}
$$

etc. etc.

45.

The examples propounded in the preceding paragraphs tell us how from the formulas (2.), (3.), (6.), (7.) § 43 the expansions of the functions $\sin^n \operatorname{am}\frac{2Kx}{\pi}$, $\frac{1}{\sin^n \operatorname{am}\frac{2Kx}{\pi}}$ are found. The quantities $A_n^{(m)}$, $B_n^{(m)}$ on which they depend can be found successively by means of the formulas (4.), (5.) from the same paragraph. But to answer the question how obtain general expressions for them, because they become too complicated to find them by induction, one has to elaborate a little more on this. For this purpose, we say the following things in advance.

The following elementary formula is known:

$$\sin\operatorname{am}(u+v)-\sin\operatorname{am}(u-v)=\frac{2\sin\operatorname{am}v\cos\operatorname{am}u\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v},$$

having integrated which with respect to u it arises:

$$(1.)\qquad \int_0^u du\,\{\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)\}=\frac{1}{k}\ln\left(\frac{1+k\sin\mathrm{am}\,u\sin\mathrm{am}\,v}{1-k\sin\mathrm{am}\,u\sin\mathrm{am}\,v}\right).$$

From Talyor's theorem it is:

$$\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)=2\left\{\frac{d\sin\mathrm{am}\,u}{du}\cdot v+\frac{d^3\sin\mathrm{am}\,u}{du^3}\cdot\frac{v^3}{\Pi 3}+\frac{d^5\sin\mathrm{am}\,u}{du^5}\cdot\frac{v^5}{\Pi 5}+\cdots\right\},$$

whence:

$$\int_0^u du\,\{\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)\}=2\left\{\sin\mathrm{am}\,u\cdot v+\frac{d^2\sin\mathrm{am}\,u}{du^2}\cdot\frac{v^3}{\Pi 3}+\frac{d^4\sin\mathrm{am}\,u}{du^4}\cdot\frac{v^5}{\Pi 5}+\cdots\right\}.$$

For, it easily becomes clear having put $u=0$ that both $\sin\mathrm{am}\,u$ and in general $\frac{d^{2m}\sin\mathrm{am}\,u}{du^{2m}}$ vanish. Hence, equation (1.), having also expanded its other side, goes over into this one:

$$(2.)\qquad \sin\mathrm{am}\,u\cdot v+\frac{d^2\sin\mathrm{am}\,u}{du^2}\cdot\frac{v^3}{\Pi 3}+\frac{d^4\sin\mathrm{am}\,u}{du^4}\cdot\frac{v^5}{\Pi 5}+\text{etc.}$$

$$=\sin\mathrm{am}\,u\sin\mathrm{am}\,v+\frac{k^2}{3}\sin^3\mathrm{am}\,u\sin^3\mathrm{am}\,v+\frac{k^4}{5}\sin^5\mathrm{am}\,u\sin^5\mathrm{am}\,v+\cdots$$

Further, having multiplied the known equations:

$$\sin\mathrm{am}(u+v)+\sin\mathrm{am}(u-v)=\frac{2\sin\mathrm{am}\,u\cos\mathrm{am}\,v\Delta\,\mathrm{am}\,v}{1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v}$$
$$\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)=\frac{2\sin\mathrm{am}\,v\cos\mathrm{am}\,u\Delta\,\mathrm{am}\,u}{1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v}$$

by each other we obtain:

$$(2.)\qquad \sin^2\mathrm{am}(u+v)-\sin^2\mathrm{am}(u-v)$$

$$=\frac{4\sin\mathrm{am}\,u\cos\mathrm{am}\,u\Delta\,\mathrm{am}\,u\cdot\sin\mathrm{am}\,v\cos\mathrm{am}\,v\Delta\,\mathrm{am}\,v}{[1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v]^2}=\frac{d\sin^2\mathrm{am}\,u\cdot d\sin^2\mathrm{am}\,v}{[1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v]^2dudv}.$$

Having executed the integration with respect to v it arises:

$$\int_0^v dv \left\{ \sin^2 \operatorname{am}(u+v) - \sin^2 \operatorname{am}(u-v) \right\}$$

$$= \frac{2 \sin \operatorname{am} u \cos \operatorname{am} u \Delta \operatorname{am} u \cdot \sin^2 \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v} = \frac{\sin^2 \operatorname{am} v \cdot d \sin^2 \operatorname{am} u}{(1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v) du}$$

Having integrated this equation once again but with respect to the other element u we obtain:

$$(4.) \qquad \int_0^u du \int_0^v dv \left\{ \sin^2 \operatorname{am}(u+v) - \sin^2 \operatorname{am}(u-v) \right\} = -\frac{1}{k^2} \ln(1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v).$$

From Taylor's theorem it is:

$$\sin^2 \operatorname{am}(u+v) - \sin^2 \operatorname{am}(u-v)$$

$$2 \left\{ \frac{d \sin^2 \operatorname{am} u}{du} \cdot v + \frac{d^3 \sin^2 \operatorname{am} u}{du^3} \cdot \frac{v^3}{\Pi 3} + \frac{d^5 \sin^2 \operatorname{am} u}{du^5} \cdot \frac{v^5}{\Pi 5} + \cdots \right\},$$

whence:

$$\int_0^v dv \left\{ \sin^2 \operatorname{am}(u+v) - \sin^2 \operatorname{am}(u-v) \right\}$$

$$2 \left\{ \frac{d \sin^2 \operatorname{am} u}{du} \cdot \frac{v^2}{\Pi 2} + \frac{d^3 \sin^2 \operatorname{am} u}{du^3} \cdot \frac{v^4}{\Pi 4} + \frac{d^5 \sin^2 \operatorname{am} u}{du^5} \cdot \frac{v^6}{\Pi 6} + \cdots \right\}$$

$$\int_0^u du \int_0^v dv \left\{ \sin^2 \operatorname{am}(u+v) - \sin^2 \operatorname{am}(u-v) \right\}$$

$$2 \left\{ \sin^2 \operatorname{am} u \cdot \frac{v^2}{\Pi 2} + \frac{d^2 \sin^2 \operatorname{am} u}{du^2} \cdot \frac{v^4}{\Pi 4} + \frac{d^4 \sin^2 \operatorname{am} u}{du^4} \cdot \frac{v^6}{\Pi 6} + \cdots \right\} - 2 \left\{ U^{(2)} \frac{v^4}{\Pi 4} + U^{(4)} \frac{v^6}{\Pi 6} + \cdots \right\}$$

if by the character $U^{(2m)}$ we denote the value of the expression $\frac{d^{2m} \sin^2 \operatorname{am} u}{du^{2m}}$ which it obtains for $u = 0$. Hence, equation (4.), having also expanded its other side, goes over into this one:

$$(5.)\quad \sin^2 \operatorname{am} u \cdot \frac{v^2}{\Pi 2} + \frac{d^2 \sin^2 \operatorname{am} u}{du^2} \cdot \frac{v^4}{\Pi 4} + \frac{d^4 \sin^2 \operatorname{am} u}{du^4} \cdot \frac{v^6}{\Pi 6} + \cdots - \left\{ U^{(2)} \frac{v^4}{\Pi 4} + U^{(4)} \frac{v^6}{\Pi 6} + \cdots \right\}$$

$$= \frac{1}{2} \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v + \frac{k^2}{4} \sin^4 \operatorname{am} u \sin^4 \operatorname{am} v + \frac{k^4}{6} \sin^6 \operatorname{am} u \sin^6 \operatorname{am} v + \cdots$$

Having prepared these thing in the right manner put

$$u = \sin \operatorname{am} u + R_1 \sin^3 \operatorname{am} u + R_2 \sin^5 \operatorname{am} u + +R_3 \sin^7 \operatorname{am} u + \cdots,$$

and in general

$$\begin{aligned} u^n &= [\sin \operatorname{am} u + R_1 \sin^3 \operatorname{am} u + R_2 \sin^5 \operatorname{am} u + +R_3 \sin^7 \operatorname{am} u + \cdots]^n \\ &= \sin^n \operatorname{am} u + R_1^{(n)} \sin^{n+2} \operatorname{am} u + R_3^{(n)} \sin^{n+4} \operatorname{am} u + R_5^{(n)} \sin^{n+6} \operatorname{am} u + \cdots; \end{aligned}$$

further, from the inversion of the series:

$$\sin \operatorname{am} u + R_1 \sin^3 \operatorname{am} u + R_2 \sin^5 \operatorname{am} u + +R_3 \sin^7 \operatorname{am} u + \cdots$$

let this one arise:

$$\sin \operatorname{am} u = u + S_1 u^3 + S_2 u^5 + S_3 u^7 + \cdots,$$

and let again be:

$$\sin^n \operatorname{am} u = [u + S_1 u^3 + S_2 u^5 + S_3 u^7 + \cdots]^n = u^n + S_1^{(n)} u^{n+2} + S_2^{(n)} u^{n+4} + S_3^{(n)} u^{n+6} + \cdots,$$

Now, from equation (2.):

$$\begin{aligned} &\sin \sin \operatorname{am} u \cdot v + \frac{d^2 \sin \operatorname{am} u}{du^2} \cdot \frac{v^3}{\Pi 3} + \frac{d^4 \sin \operatorname{am} u}{du^4} \cdot \frac{v^5}{\Pi 5} + \cdots \\ &= \sin \operatorname{am} u \sin \operatorname{am} v + \frac{k^2}{3} \sin^3 \operatorname{am} u \sin^3 \operatorname{am} v + \frac{k^4}{5} \sin^5 \operatorname{am} u \sin^5 \operatorname{am} v + \cdots, \end{aligned}$$

having expanded v, v^3, v^5 etc. into a series of powers of $\sin \operatorname{am} v$ and having compared the coefficients of $\sin^{2n+1} \operatorname{am} v$ on both sides of the equation, it arises:

$$(6.)\qquad \frac{k^{2n}\sin^{2n+1}\operatorname{am}}{2n+1}$$
$$= R_n^{(1)}\sin\operatorname{am} + R_{n-1}^{(3)}\frac{d^2\sin\operatorname{am}}{\Pi 3\cdot du^2} + R_{n-2}^{(5)}\frac{d^4\sin\operatorname{am}}{\Pi 5\cdot du^4} + \cdots \frac{d^{2n}\sin\operatorname{am} u}{\Pi(2n+1)du^{2n}}.$$

In the same way it arises from formula (5.):

$$(7.)\qquad \frac{k^{2n-2}\sin^{2n}\operatorname{am}}{2n}$$
$$= R_{n-1}^{(2)}\frac{\sin^2\operatorname{am} u}{\Pi 2} + R_{n-2}^{(4)}\frac{d^2\sin^2\operatorname{am} u}{\Pi 4\cdot du^2} + R_{n-3}^{(6)}\frac{d^4\sin^2\operatorname{am} u}{\Pi 6\cdot du^4} + \cdots + \frac{d^{2n-2}\sin^2\operatorname{am} u}{\Pi(2n)\cdot du^{2n-2}}$$
$$-\left\{\frac{R_{n-2}^{(4)}}{3\cdot 4} + \frac{R_{n-3}^{(6)}}{5\cdot 6}S_1^{(2)} + \frac{R_{n-4}^{(8)}}{7\cdot 8}S_2^{(2)} + \cdots + \frac{S_{n-2}^{(2)}}{(2n-1)\cdot 2n}\right\}$$

From (6.), (7.) having changed u into $u+iK'$ it follows:

$$(8.)\qquad \frac{1}{(2n+1)\sin^{2n+1}\operatorname{am} u}$$
$$= \frac{R_n^{(1)}}{\sin\operatorname{am} u} + \frac{R_{n-1}^{(3)}}{\Pi 3}\cdot\frac{d^2}{du^2}\frac{1}{\sin\operatorname{am} u} + \frac{R_{n-2}^{(5)}}{\Pi 5}\cdot\frac{d^4}{du^4}\frac{1}{\sin\operatorname{am} u} + \cdots + \frac{1}{\Pi(2n+1)}\cdot\frac{d^{2n}}{du^{2n}}\frac{1}{\sin\operatorname{am} u}$$

$$(9.)\qquad \frac{1}{(2n)\sin^{2n}\operatorname{am} u}$$
$$= \frac{R_{n-1}^{(2)}}{\Pi 2\cdot\sin^2\operatorname{am} u} + \frac{R_{n-2}^{(4)}}{\Pi 4}\cdot\frac{d^2}{du^2}\frac{1}{\sin^2\operatorname{am} u} + \frac{R_{n-3}^{(6)}}{\Pi 6}\cdot\frac{d^4}{du^4}\frac{1}{\sin^2\operatorname{am} u} + \cdots + \frac{1}{\Pi(2n)}\cdot\frac{d^{2n-2}}{du^{2n-2}}\frac{1}{\sin^2\operatorname{am} u}$$
$$-k^2\left\{\frac{R_{n-2}^{(4)}}{3\cdot 4} + \frac{R_{n-3}^{(6)}}{5\cdot 6}S_1^{(2)} + \frac{R_{n-4}^{(8)}}{7\cdot 8}S_2^{(2)} + \cdots + \frac{S_{n-2}^{(2)}}{(2n-1)2n}\right\}.$$

These are the general formulas we are looking for by means of which $\sin^n\operatorname{am} u$, $\frac{1}{\sin^n\operatorname{am} u}$ are found from $\sin\operatorname{am} u$, $\sin^2\operatorname{am} u$, $\frac{1}{\sin\operatorname{am} u}$, $\frac{1}{\sin^2\operatorname{am} u}$ and its differentials.

At this occasion I remark, if vice versa $\sin\operatorname{am} v$, $\sin^2\operatorname{am} v$, $\sin^3\operatorname{am} v$, etc. are expanded into a power series in v, that from the formulas (2.), (5.) it is found:

$$(10.)\qquad \frac{d^{2n}\sin\operatorname{am} u}{\Pi(2n+1)du^{2n+1}}$$

$$= S_n^{(1)} \sin\operatorname{am} u + \frac{k^2}{3} S_{n-1}^{(3)} \sin^3 \operatorname{am} u + \frac{k^4}{5} S_{n-2}^{(5)} \sin^5 \operatorname{am} + \cdots \frac{k^{2n}}{2n+1} \sin^{2n+1} \sin^{2n+1} \operatorname{am} u$$

$$(11.) \qquad \frac{d^{2n} \sin^2 \operatorname{am} u}{\Pi(2n+2) du^{2n}} - \frac{S_{n-1}^{(2)}}{(2n+1)(2n+2)}$$

$$= \frac{1}{2} S_n^{(2)} \sin^2 \operatorname{am} u + \frac{k^2}{4} S_{n-1}^{(4)} \sin^4 \operatorname{am} u + \frac{k^3}{6} S_{n-2}^{(6)} \sin^6 \operatorname{am} u + \cdots + \frac{k^{2n}}{2n+2} \sin^{2n+2} \operatorname{am} u.$$

Finally, some things concerning the invention of $R_m^{(n)}$, $S_m^{(n)}$ are to be added. Having put $\sin\operatorname{am} u = y$, it is from the propounded definition:

$$u = \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = y + R_1 y^3 + R_2 y^5 + R_3 y^7 + \cdots$$

or:

$$\frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 1 + 3R_1 y^2 + 5R_2 y^4 + 7R_3 y^6 + \cdots;$$

hence;

$$3R_1 = \frac{1+k^2}{2}, \quad 5R_2 = \frac{1\cdot 3}{2\cdot 4} + \frac{1}{2}\cdot\frac{1}{2}k^2 + \frac{1\cdot 3}{2\cdot 4}k^4$$
$$7R_3 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2}k^2 + \frac{1}{2}\cdot\frac{1\cdot 3}{2\cdot 4}k^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6$$
$$9R_4 = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{2}k^2 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1\cdot 3}{2\cdot 4}k^4 + \frac{1}{2}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6 + \frac{1\cdot 3\cdot 5\cdot 7}{\cdot 2\cdot 4\cdot 6\cdot 8}k^8$$

etc. etc.

or also:

$$
\begin{aligned}
3R_1 &= \frac{1}{2} && \cdot (1+k^2) \\
5R_2 &= \frac{1\cdot 3}{2\cdot 4} && \cdot (1+k^2)^2 - \frac{1}{2} && \cdot k^2 \\
7R_3 &= \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} && \cdot (1+k^2)^3 - \frac{1\cdot 3}{2\cdot 2} && \cdot k^2(1+k^2) \\
9R_4 &= \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} && \cdot (1+k^2)^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 2} && \cdot k^2(1+k^2)^2 + \frac{1\cdot 3}{2\cdot 4}k^4 \\
11R_5 &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10} && \cdot (1+k^2)^5 - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 2} && \cdot k^2(1+k^2)^3 + \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 4}k^4(1+k^2) \\
13R_6 &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} && \cdot (1+k^2)^6 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 2} && \cdot k^2(1+k^2)^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 2\cdot 4}k^4(1+k^2)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6
\end{aligned}
$$

or also:

$$
\begin{aligned}
3R_1 &= 1 - \frac{1}{2}\cdot 1\cdot k'^2 \\
5R_2 &= 1 - \frac{1}{2}\cdot 2\cdot k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 1\cdot k'^4 \\
7R_3 &= 1 - \frac{1}{2}\cdot 3\cdot k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 3\cdot k'^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 1\cdot k'^6 \\
9R_4 &= 1 - \frac{1}{2}\cdot 4\cdot k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 6\cdot k'^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 4\cdot k'^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}k'^8 \\
&\qquad\qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}
$$

or finally:

$$
\begin{aligned}
3R_1 &= k^2 + \frac{1}{2}\cdot k'^2 \\
5R_2 &= k^4 + \frac{1}{2}\cdot 2k^2k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot k'^4 \\
7R_3 &= k^6 + \frac{1}{2}\cdot 3k^4k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 3k^2\cdot k'^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot k'^6 \\
9R_4 &= k^8 + \frac{1}{2}\cdot 4k^6k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 6k^2\cdot k'^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 4k^2k'^6 + \frac{1\cdot 3\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}k'^8 \\
&\qquad \text{etc.} \qquad\qquad\qquad \text{etc.}
\end{aligned}
$$

From these four ways to express the quantities R_m the second way yields a sufficiently memorable and beautiful representation of them if we introduce the quantities:

$$r = \frac{1+k^2}{2k}.$$

For the sake of an example it is:

$$\frac{13R_6}{k^6} = \frac{1\cdot 3\cdots 11}{1\cdot 2\cdots 6}r^6 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 2}r^4 + \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 2\cdot 4}r^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},$$

having integrated which expression 6 times with respect to r, we obtain:

$$13\int^6 \frac{R_6 dr^6}{k^6} = \frac{r^{12}}{2\cdot 4\cdots 12} - \frac{r^{10}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 2} + \frac{r^8}{2\cdot 4\cdot 6\cdot 8\cdot 2\cdot 4} - \frac{r^6}{2\cdot 4\cdot 6\cdot 2\cdot 4\cdot 6} + C'r^4 + C''r^2 + C''',$$

C', C'', C''' denoting arbitrary constants. Having conveniently determined them it arises:

$$13\int^6 \frac{R_6 dr^6}{k^6} = \frac{(r^2-1)^6}{2^6\cdot \Pi 6},$$

whence vice versa:

$$13R_6 = \frac{k^6 d(r^2-1)^6}{2^6\cdot \Pi 6\cdot dr^6};$$

and in the same way it is obtained in general:

$$(12.) \qquad (2m+1)R_m = \frac{k^m d^m (r^2-1)^m}{2^m\cdot \Pi m\cdot dr^m}.$$

Confer the short commentary (*Crelle Journal II. p.223*) entitled:

"Ueber eine besondere Gattungen algebraischer Functionen, die aus der Entwicklung der Function $(1-2xz+z^2)^{-\frac{1}{2}}$ entstehn."

Having found the quantities R_m by means of known algorithms one has to find quantities $R_m^{(n)}$, $S_m^{(n)}$ that:

$$[1 + R_1 x + R_2 x^2 + R_3 x^3 + \cdots]^n = 1 + R_1^{(n)} x + R_2^{(n)} x^2 + R_3^{(n)} x^3 + \cdots,$$

further, if it is put:

$$y = x[1 + R_1 x^2 + R_2 x^4 + R_3 x^6 + \cdots],$$

let:

$$x^n = y^n[1 + S_1^{(n)} y^2 + s_2^{(n)} y^4 + S_3^{(n)} y^6 + \cdots];$$

these agree with the definition of the quantities $R_m^{n)}$, $S_m^{(n)}$ propounded above. But, having put:

$$\varphi(x) = 1 + R_1 x + R_2 x^2 + R_3 x^3 + \cdots ,$$

ii is from a theorem found by McLaurin and Lagrange:

$$\begin{aligned} R_m^{(n)} &= \frac{d^m[\varphi x]^n}{\Pi m \cdot dx^m} \\ S_m^{(n)} &= \frac{n}{2m+n} \cdot \frac{d^m[\varphi x]^{-(2m+n)}}{\Pi m \cdot dx^m}, \end{aligned}$$

if one puts $x = 0$ after the differentiation.

46.

By means of the formulas (6.), (7.), (8.), (9.), § 45 we obtain the following general expansions:

$$\text{(1.)} \qquad \frac{\left(\frac{2kK}{\pi}\right)^{2n+1} \sin^{2n+1} \operatorname{am} \frac{2Kx}{\pi}}{2n+1}$$

$$\begin{aligned} &= 4\left\{ R_n^{(1)} \left(\frac{2K}{\pi}\right)^{2n} - \frac{R_{n-1}^{(3)}}{\Pi 3} \left(\frac{2K}{\pi}\right)^{2n-2} + \frac{R_{n-2}^{(5)}}{\Pi 5} \left(\frac{2K}{\pi}\right)^{2n-4} - \cdots + \frac{(-1)^n}{\Pi(2n+1)} \right\} \frac{\sqrt{q} \sin x}{1-q} \\ &+ 4\left\{ R_n^{(1)} \left(\frac{2K}{\pi}\right)^{2n} - \frac{3^2 R_{n-1}^{(3)}}{\Pi 3} \left(\frac{2K}{\pi}\right)^{2n-2} + \frac{3^4 R_{n-2}^{(5)}}{\Pi 5} \left(\frac{2K}{\pi}\right)^{2n-4} - \cdots + \frac{(-1)^n 3^{2n}}{\Pi(2n+1)} \right\} \frac{\sqrt{q^3} \sin 3x}{1-q^3} \\ &= 4\left\{ R_n^{(1)} \left(\frac{2K}{\pi}\right)^{2n} - \frac{5^2 R_{n-1}^{(3)}}{\Pi 3} \left(\frac{2K}{\pi}\right)^{2n-2} + \frac{5^4 R_{n-2}^{(5)}}{\Pi 5} \left(\frac{2K}{\pi}\right)^{2n-4} - \cdots + \frac{(-1)^n 5^{2n}}{\Pi(2n+1)} \right\} \frac{\sqrt{q^5} \sin 5x}{1-q^5} \\ &+ \cdots \end{aligned}$$

$$(2.)\qquad \frac{\left(\frac{2kK}{\pi}\right)^{2n}\sin^{2n}\operatorname{am}\frac{2Kx}{\pi}}{2n}$$

$$= \frac{R_{n-1}^{(2)}}{\Pi 2}\left(\frac{2K}{\pi}\right)^{2n-1}\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right) - k^2\left(\frac{2K}{\pi}\right)^{2n}\left\{\frac{R_{n-2}^{(4)}}{3\cdot 4}+\frac{R_{n-3}^{(6)}}{5\cdot 6}S_1^{(2)}+\frac{R_{n-4}^{(8)}}{7\cdot 8}S_2^{(2)}+\cdots+\frac{S_{n-2}^{(2)}}{(2n-1)2n}\right\}$$

$$-4\left\{\frac{2R_{n-1}^{(2)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{2^3R_{n-3}^{(4)}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^{n-1}2^{2n-1}}{\Pi 2n}\right\}\frac{q\cos 2x}{1-q^2}$$

$$-4\left\{\frac{4R_{n-1}^{(2)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{4^3R_{n-3}^{(4)}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^{n-1}4^{2n-1}}{\Pi 2n}\right\}\frac{q^2\cos 4x}{1-q^4}$$

$$-4\left\{\frac{6R_{n-1}^{(2)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{6^3R_{n-3}^{(4)}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^{n-1}6^{2n-1}}{\Pi 2n}\right\}\frac{q^3\cos 6x}{1-q^6}$$

$$-\cdots$$

$$(3.)\qquad \frac{\left(\frac{2K}{\pi}\right)^{2n+1}}{(2n+1)\sin^{2n+1}\operatorname{am}\frac{2Kx}{\pi}}$$

$$= \frac{R_n^{(1)}\left(\frac{2K}{\pi}\right)^{2n}}{\sin x}+\frac{R_{n-1}^{(3)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}\cdot\frac{d^2}{dx^2}\frac{1}{\sin x}+\cdots+\frac{1}{\Pi(2n+1)}\cdot\frac{d^{2n}}{dx^{2n}}\frac{1}{\sin x}$$

$$+4\left\{R_n^{(1)}\left(\frac{2K}{\pi}\right)^{2n}-\frac{R_{n-1}^{(3)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}+\cdots+\frac{(-1)^n}{\Pi(2n+1)}\right\}\frac{q\sin x}{1-q}$$

$$+4\left\{R_n^{(1)}\left(\frac{2K}{\pi}\right)^{2n}-\frac{3^2R_{n-1}^{(3)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}+\cdots+\frac{(-1)^n3^{2n}}{\Pi(2n+1)}\right\}\frac{q^3\sin 3x}{1-q^3}$$

$$+4\left\{R_n^{(1)}\left(\frac{2K}{\pi}\right)^{2n}-\frac{5^2R_{n-1}^{(3)}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}+\cdots+\frac{(-1)^n5^{2n}}{\Pi(2n+1)}\right\}\frac{q^5\sin 5x}{1-q^5}$$

$$+\cdots$$

$$\text{(4.)}\qquad \frac{\left(\frac{2K}{\pi}\right)^{2n}}{2n\cdot\sin^{2n}\operatorname{am}\frac{2Kx}{\pi}}$$

$$= \frac{1}{2}R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-1}\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right) - k^2\left(\frac{2K}{\pi}\right)^{2n}\left\{\frac{1}{3\cdot 4}R^{(4)}_{n-2}+\frac{1}{5\cdot 6}R^{(6)}_{n-3}S^{(2)}_1+\frac{1}{7\cdot 8}R^{(8)}_{n-4}S^{(2)}_2+\cdots+\frac{1}{(2n-1)2n}S^{(2)}_{n-2}\right\}$$

$$+\frac{R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}+\frac{R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}\cdot\frac{d^2}{dx^2}\frac{1}{\sin^2 x}+\cdots+\frac{1}{\Pi 2n}\cdot\frac{d^{2n-2}}{dx^{2n-2}}\frac{1}{\sin^2 x}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{2^3R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^n 2^{2n-1}}{\Pi 2n}\right\}\frac{q^2\cos 2x}{1-q^2}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{4^3R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^n 4^{2n-1}}{\Pi 2n}\right\}\frac{q^4\cos 4x}{1-q^4}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{6^3R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^n 6^{2n-1}}{\Pi 2n}\right\}\frac{q^6\cos 6x}{1-q^6}$$

$$-\cdots$$

From the formulas (6.), (7.), (8.), (9.) § 45 one can deduce other which involve the functions $\cos\operatorname{am} u$, $\tan\operatorname{am} u$, $\Delta\operatorname{am} u$ instead of $\sin\operatorname{am} u$. For, from the formula:

$$\sin\operatorname{am}\left(k'u, \frac{ik}{k'}\right) = \cos\operatorname{coam} u,$$

whence also:

$$\sin\operatorname{am}\left(k'(K-u), \frac{ik}{k'}\right) = \cos\operatorname{am} u,$$

we see that in the propounded formulas, if one puts $\frac{ik}{k'}$ instead of k and $k'(K-u)$ instead of u, $\sin\operatorname{am} u$ goes over into $\cos\operatorname{am} u$, whence one finds similar formulas which correspond to $\cos\operatorname{am} u$. Further, from the equation:

$$\sin\operatorname{am} iu = i\tan\operatorname{am}(u, k')$$

it is clear that at the same time one can change u into iu, k into k', $\sin\operatorname{am} u$ into $i\tan\operatorname{am} u$; hence, we find formulas for $\tan\operatorname{am} u$. Finally, from these, because

$$\cot\operatorname{am}(u+iK') = -i\Delta\operatorname{am} u,$$

one can find formulas for $\Delta\,\mathrm{am}\,u$ corresponding to the formulas (6.), (7.), (8.), (9.). Having found these by means of a similar method from the expansion of the functions:

$$\begin{array}{llll} \cos\mathrm{am}\,\dfrac{2Kx}{\pi}, & \cos^2\mathrm{am}\,\dfrac{2Kx}{\pi}, & \Delta\,\mathrm{am}\,\dfrac{3Kx}{\pi}, & \Delta^2\,\mathrm{am}\,\dfrac{2Kx}{\pi} \\ \dfrac{1}{\cos\mathrm{am}\,\frac{2Kx}{\pi}}, & \dfrac{1}{\cos^2\mathrm{am}\,\frac{2Kx}{\pi}}, & \dfrac{1}{\Delta\,\mathrm{am}\,\frac{2Kx}{\pi}}, & \dfrac{1}{\Delta^2\,\mathrm{am}\,\frac{2Kx}{\pi}} \end{array}$$

propounded by us one deduces general expansions of the functions:

$$\cos^n\mathrm{am}\,\frac{2Kx}{\pi}, \qquad \Delta^n\,\mathrm{am}\,\frac{2Kx}{\pi}.$$

It shall be sufficient to have mentioned these things.

We obtain extraordinary transformations of the series into which we expanded the elliptic functions after having put ix instead of x and applied the formulas which we gave for the reduction of an imaginary argument to a real argument in the first foundations. But because those are easy to obtain we do not want to treat this subject here any longer.

2.3 The second kind of elliptic Functions is expanded into series

47.

Having integrated the integral formula exhibited above in § 41 (1.):

$$\left(\frac{2kK}{\pi}\right)^2\sin^2\mathrm{am}\,\frac{2Kx}{\pi} = \frac{2K}{\pi}\frac{2K}{\pi} - 4\left\{\frac{2q\cos 2x}{1-q^2} + \frac{4q^2\cos 4x}{1-q^4} + \frac{6q^3\cos 6x}{1-q^6} + \cdots\right\}$$

from $x = 0$ to $x = x$ it arises:

$$\left(\frac{2kK}{\pi}\right)^2\int\limits_0^x \sin^2\mathrm{am}\,\frac{2Kx}{\pi}$$

$$= \left\{\frac{2K}{\pi}\frac{2k}{\pi} - \frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi}\right\}x - 4\left\{\frac{q\sin 2x}{1-q^2} + \frac{q\sin 4x}{1-q^4} + \frac{q^3\sin 6x}{1-q^6} + \frac{q^4\sin 8x}{1-q^8} + \cdots\right\}.$$

In the following, let us denote by the character $\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)$ the expression:

$$(1.)\qquad \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)=\frac{2Kx}{\pi}\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right)-\left(\frac{2kK}{\pi}\right)^2\int\limits_0^x \sin^2\operatorname{am}\frac{2Kx}{\pi}dx$$

$$=4\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^2\sin 4x}{1-q^4}+\frac{q^3\sin 6x}{1-q^6}+\frac{q^4\sin 8x}{1-q^8}+\cdots\right\}.$$

From Legendre's notations having put $\frac{2Kx}{\pi}=u$, $\varphi=\operatorname{am}u$ it will be:

$$(2.)\qquad Z(u)=\frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}}.$$

It is convenient to introduce the function $Z(u)$ instead of $E(\varphi)$ into the analysis of elliptic functions; moreover, it is easy to reduce it to the functions used by Legendre by means of formula (2.). We want to sketch a little bit, how from the expansion of the function Z which formula (1.) yields it is possible to derive many of its properties even though they are known.

In (1.) change x to $x+\frac{\pi}{2}$, it arises:

$$\frac{2K}{\pi}Z\left(\frac{2kx}{\pi}+K\right)=-4\left\{\frac{q\sin 2x}{1-q^2}-\frac{q^2\sin 4x}{1-q^4}+\frac{q^3\sin 6x}{1-q^6}-\cdots\right\},$$

whence:

$$\frac{2k}{\pi}Z\left(\frac{2Kx}{\pi}\right)-\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}+K\right)=8\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^3\sin 6x}{1-q^6}+\frac{q^5\sin 10x}{1-q^{10}}+\cdots\right\}.$$

Further, in (1.) change x to $2x$, q to q^2, and at the same time k to $k^{(2)}$, K to $K^{(2)}$, it arises:

$$\frac{2K^{(2)}}{\pi}Z\left(\frac{4K^{(2)}}{\pi},k^{(2)}\right)=4\left\{\frac{q^2\sin 4x}{1-q^4}+\frac{q^4\sin 8x}{1-q^8}+\frac{q^6\sin 12x}{1-q^{12}}+\cdots\right\},$$

whence:

$$\frac{2K}{\pi}Z\left(\frac{2kx}{\pi}\right)-\frac{2K^{(2)}}{\pi}Z\left(\frac{4K^{(2)}}{\pi},k^{(2)}\right)=\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^3\sin 6x}{1-q^6}+\frac{q^5\sin 10x}{1-q^{10}}+\cdots\right\}.$$

But above we found:

$$\frac{2kK}{\pi}\sin\mathrm{am}\frac{2kX}{\pi}=\left\{\frac{\sqrt{q}\sin x}{1-q}+\frac{\sqrt{q^3}\sin 3x}{1-q^3}+\frac{\sqrt{q^5}\sin 5x}{1-q^5}+\cdots\right\},$$

whence having changed q to q^2, x to $2x$:

$$\frac{2k^{(2)}K^{(2)}}{\pi}\sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi},k^{(2)}\right)=4\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^3\sin 6x}{1-q^6}+\frac{q^5\sin 10x}{1-q^{10}}+\cdots\right\}.$$

Hence, it follows:

$$(3.)\quad \frac{2K}{\pi}\left\{Z\left(\frac{2Kx}{\pi}\right)-Z\left(\frac{2Kx}{\pi}+K\right)\right\}=\frac{4k^{(2)}K^{(2)}}{\pi}\sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi},k^{(2)}\right)$$

$$(4.)\quad \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)-\frac{2K^{(2)}}{\pi}Z\left(\frac{4K^{(2)}x}{\pi},k^{(2)}\right)=\frac{2k^{(2)}K^{(2)}}{\pi}\sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi},k^{(2)}\right)$$

$$(5.)\quad \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)+\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}+K\right)-\frac{4K^{(2)}}{\pi}\left(\frac{4K^{(2)}x}{\pi},k^{(2)}\right)=0.$$

In these formulas, of which (4.), (5.) yield the transformation of the function Z of second order, it is:

$$k^{(2)}=\frac{1-k'}{1+k'},\quad K^{(2)}=\frac{1+k'}{2}\cdot K,\quad \sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi},k^{(2)}\right)=(1+k')\sin\mathrm{am}\frac{2Kx}{\pi}\cdot\sin\mathrm{coam}\frac{2Kx}{\pi},$$

as it is known about the transformation of second order propounded by Legendre. Hence, formula (3.) can also be represented this way having put $u=\frac{2Kx}{\pi}$:

$$(6.)\qquad Z(u)-Z(u+K)=k^2\sin\mathrm{am}\,u\sin\mathrm{coam}\,u.$$

For the sake of brevity let us put $\mathrm{am}\left(\frac{2mK^{(m)}x}{\pi},k^{(m)}\right)=\varphi^{(m)}$, from formula (4.), having successively put $k^{(2)}$,$k^{(4)}$, $k^{(8)}$, $k^{(16)}\cdots$ instead of k; $2x$, $4x$, $8x\cdots$, instead of x, it arises:

$$(7.)\quad K\cdot Z(u)=F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)=k^{(2)}K^{(2)}\sin\varphi^{(2)}+k^{(4)}K^{(4)}\sin\varphi^{(4)}+k^{(8)}K^{(8)}\sin\varphi^{(8)}+\cdots,$$

which formula Legendre gave.

In similar manner from formula § 41:

$$\frac{2K}{\pi}\frac{2K}{\pi}-\frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi}=8\left\{\frac{q}{(1-q)^2}+\frac{q^3}{(1-q^3)^2}+\frac{q^5}{(1-q^5)^2}+\frac{q^7}{(1-q^7)^2}+\cdots\right\},$$

which can also be expanded this way:

$$\frac{2K}{\pi}\frac{2K}{\pi}-\frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi}=8\left\{\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}+\frac{3q^3}{1-q^6}+\frac{4q^4}{1-q^8}+\cdots\right\},$$

having compared it to this one we found above:

$$\left(\frac{2kK}{\pi}\right)^2=16\left\{\frac{q}{1-q^2}+\frac{3q^3}{1-q^5}+\frac{5q^5}{1-q^{10}}+\frac{7q^7}{1-q^{14}}+\cdots\right\},$$

it arises:

$$(8.)\qquad 2K(K-E^{\mathrm{I}})=(kK)^2+2(k^{(2)}K^{(2)})^2+4(k^{(4)}K^{(4)})^2+8(k^{(8)}K^{(8)})^2+\cdots,$$

which agrees with that one Gauß gave in his paper *Determinatio attractionis* etc. § 17.

48.

By means of the same method, by which in § 41 we found the expansion of the expression $\left(\frac{2kK}{\pi}\right)^2\sin^2\operatorname{am}\frac{2Kx}{\pi}$, let us investigate how to expand the expression $\left\{\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)\right\}^2$ into a series. Let us prove:

$$\begin{aligned}\left(\frac{2K}{\pi}\right)^2 Z\left(\frac{2Kx}{\pi}\right)Z\left(\frac{2Kx}{\pi}\right)&=16\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^2\sin 4x}{1-q^4}+\frac{q^3\sin 6x}{1-q^6}+\cdots\right\}^2\\&=8\left\{A+A'\cos 2x+A''\cos 4x+A'''\cos 6x+\cdots\right\},\end{aligned}$$

which expression we see to take the propounded form if instead of $2\sin 2mx \sin m'x$ one puts $\cos 2(m-m')x - \cos 2(m+m'x$ everywhere. At first it is:

$$A = \frac{q^2}{(1-q^2)^2} + \frac{q^4}{(1-q^4)^2} + \frac{q^6}{(1-q^6)^2} + \frac{q^8}{(1-q^8)^2} + \cdots$$

After this, in general we obtain: $A^{(n)} = 2B^{(n)} - C^{(n)}$, if it is put:

$$\begin{aligned} B^{(n)} &= \frac{q^{n+2}}{(1-q^2)(1-q^{2n+2})} + \frac{q^{n+4}}{(1-q^4)(1-q^{2n+4})} + \frac{q^{n+6}}{(1-q^6)(1-q^{2n+6})} + \cdots \\ C^{(n)} &= \frac{q^n}{(1-q^2)(1-q^{2n-2})} + \frac{q^n}{(1-q^4)(1-q^{2n-4})} + \cdots + \frac{q^n}{(1-q^{2n-2})(1-q^2)} + \cdots \end{aligned}$$

In the single terms of these expressions respectively put:

$$\begin{aligned} \frac{q^{m+n}}{(1-q^m)(1-q^{2n+m})} &= \frac{q^n}{1-q^{2n}} \left\{ \frac{q^m}{1-q^m} - \frac{q^{2n+m}}{1-q^{2n+m}} \right\} \\ \frac{q^n}{(1-q^n)(1-q^{2n-m})} &= \frac{q^n}{1-q^{2n}} \left\{ \frac{q^m}{1-q^m} + \frac{q^{2n-m}}{1-q^{2n-m}} + 1 \right\}, \end{aligned}$$

it arises:

$$\begin{aligned} B^{(n)} &= \frac{q^n}{1-q^{2n}} \left\{ \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots \right\} \\ &\quad - \frac{q^n}{1-q^{2n}} \left\{ \frac{q^{2n+2}}{1-q^{2n+2}} + \frac{q^{2n+4}}{1-q^{2n+4}} + \frac{q^{2n+6}}{1-q^{2n+6}} + \cdots \right\} \\ &= \frac{q^n}{1-q^{2n}} \left\{ \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots + \frac{q^{2n}}{1-q^{2n}} \right\} \\ C^{(n)} &= \frac{(n-1)q^n}{1-q^{2n}} + \frac{2q^n}{1-q^{2n}} \left\{ \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots + \frac{q^{2n-2}}{1-q^{2n-2}} \right\}; \end{aligned}$$

hence:

$$A^{(n)} = 2B^{(n)} - C^{(n)} = -\frac{(n-1)q^n}{1-q^{2n}} + \frac{2q^{3n}}{(1-q^{2n})^2} = -\frac{nq^n}{1-q^{2n}} + \frac{q^n(1+q^{2n})}{(1-q^{2n})^2};$$

Having collected all these, one finds the sought expansion:

$$(1.)\quad \left(\frac{2K}{\pi}\right)^2 Z\left(\frac{2Kx}{\pi}\right) Z\left(\frac{2Kx}{\pi}\right) = 8A - 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\}$$
$$+8\left\{\frac{q(1+q^2)\cos 2x}{(1-q^2)^2} + \frac{q^2(1+q^4)\cos 4x}{(1-q^4)^2} + \frac{q^3(1+q^6)\cos 6x}{(1-q^6)^2} + \cdots\right\}.$$

Because $A = \frac{q^2}{(1-q^2)^2} + \frac{q^4}{(1-q^4)^2} + \frac{q^6}{(1-q^6)^2} + \cdots$ can also be expanded this way:

$$A = \frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \frac{4q^8}{1-q^8} + \cdots,$$

we find from § 42 (6.):

$$(2.)\qquad 8A = \frac{(2-k^2)\left(\frac{2k}{\pi}\right)^2 - 3\frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} + 1}{3}.$$

Further, it is known that:

$$8A = \frac{2}{\pi}\cdot\left(\frac{2K}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} Z\left(\frac{2Kx}{\pi}\right) Z\left(\frac{2Kx}{\pi}\right) dx;$$

for, having integrated equation (1.) from $x=0$ to $x=\frac{\pi}{2}$, all terms except the first vanish; hence, if one prefers to use Legendre's notations:

$$(3.)\qquad \int_0^{\frac{\pi}{2}} \frac{[F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)]^2}{\Delta(\varphi)} d\varphi = \frac{(2-k^2)F^{\mathrm{I}}F^{\mathrm{I}}F^{\mathrm{I}} - 3F^{\mathrm{I}}F^{\mathrm{I}}E^{\mathrm{I}} + \frac{1}{4}\pi\pi F^{\mathrm{I}}}{3},$$

which is the evaluation of a rather intricate definite integral.

2.4 Indefinite Elliptic Integrals of the third Kind are reduced to the definite case, in which the parameter is equal to the amplitude

49.

Before we get to the expansion of the elliptic integrals of the third kind into series, we want to explain some things concerning their theory using a similar notation as Legendre himself. Soon the same is propounded having used new notations.

We begin with certain known theorems on elliptic integrals of the second kind. It is:

$$\sin\operatorname{am}(u+a)+\sin\operatorname{am}(u-a)=\frac{2\sin\operatorname{am}u\cos\operatorname{am}a\Delta\operatorname{am}a}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u}$$
$$\sin\operatorname{am}(u+a)-\sin\operatorname{am}(u-a)=\frac{2\sin\operatorname{am}a\cos\operatorname{am}u\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u},$$

whence:

$$\sin^2\operatorname{am}(u+a)-\sin^2\operatorname{am}(u-a)=\frac{4\sin\operatorname{am}a\cos\operatorname{am}a\Delta\operatorname{am}a\sin\operatorname{am}u\cos\operatorname{am}u\Delta\operatorname{am}u}{[1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u]^2}$$

after having integrated which formula with respect to u it arises:

$$(1.)\qquad \int_0^u du[\sin^2\operatorname{am}(u+a)-\sin^2\operatorname{am}(u-a)]=\frac{2\sin\operatorname{am}a\cos\operatorname{am}a\Delta\operatorname{am}a\sin^2\operatorname{am}u}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u},$$

as we already found above.

Put: $\operatorname{am}u=\varphi$, $\operatorname{am}a=\alpha$, $\operatorname{am}(u+a)=\sigma$, $\operatorname{am}(u-a)=\vartheta$, it will be from Legendre's notation:

$$k^2\int_0^u du\sin^2\operatorname{am}u=F(\varphi)-E(\varphi),$$

whence also, because it is $F(\sigma)-F(\alpha)=F(\varphi)$, $F(\vartheta)+F(\alpha)=F(\varphi)$:

$$k^2\int_0^u du\sin^2\operatorname{am}(u+a)=F(\varphi)-E(\sigma)+E(\alpha)$$
$$k^2\int_0^u du\sin^2\operatorname{am}(u-a)=F(\varphi)-E(\vartheta)-E(\alpha).$$

Hence, equation (1.) goes over into this one:

$$(2.)\qquad 2E(\alpha) - [E(\sigma) - E(\vartheta)] = \frac{2k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi}{1 - k^2 \sin^2\alpha \sin^2\varphi}.$$

Having interchanged u and a, α goes over into φ, φ into $-\vartheta$, σ stays unchanged, whence it arises from (2.):

$$2E(\varphi) - [E(\sigma) + E(\vartheta)] = \frac{2k^2 \sin\varphi \cos\varphi \Delta\varphi \sin^2\alpha}{1 - k^2 \sin^2\alpha \sin^2\varphi},$$

having added which to equation (2.) it arises:

$$(3.)\qquad E(\varphi) + E(\alpha) - E(\sigma) = k^2 \sin\alpha \sin\varphi \sin\sigma,$$

which is the theorem on the addition of the function E, given by Legendre, l.c. cap IX. pag. 43. c'.

Integrals of the form:

$$\int_0^\varphi \frac{\sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)}$$

according to Legendre's classification of elliptic integrals into species constitute the *third* kind. He calls the quantity $-k^2 \sin^2\alpha$, he denotes by n, the parameter; we will in the following call the angle α itself the *parameter*. For these integrals, having multiplied equation (2.) by

$$\frac{d\varphi}{\Delta(\varphi)} = \frac{d\sigma}{\Delta(\sigma)} = \frac{d\vartheta}{\Delta(\vartheta)}$$

and having integrated from $\varphi = 0$ to $\varphi = \varphi$ having done which the limits of σ will be: $\sigma = \alpha$, $\sigma = \sigma$, the limits of ϑ will be: $\vartheta = -\alpha$, $\vartheta = \vartheta$, we find the following expression:

$$\int_0^\varphi \frac{2k^2 \sin\sigma \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)} = 2F(\varphi)E(\alpha) - \int_\alpha^\sigma \frac{E(\sigma)d\sigma}{\Delta(\sigma)} + \int_{-\alpha}^\varphi \frac{E(\vartheta)d\vartheta}{\Delta(\vartheta)}.$$

It easily becomes clear, because it is $E(-\varphi) = -E(\varphi)$, that it will be:

$$\int_0^{\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = \int_0^{-\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} \quad \text{or} \quad \int_{-\varphi}^{\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = 0,$$

whence, because it is:

$$\int_{\alpha}^{\sigma} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} = \int_0^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} - \int_0^{\alpha} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}$$
$$\int_{-\alpha}^{\vartheta} \frac{E(\vartheta)d\vartheta}{\Delta(\vartheta)} = \int_0^{\vartheta} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} - \int_0^{-\alpha} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} = \int_0^{\vartheta} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} - \int_0^{-\alpha} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

we now obtain a new and memorable

Theorem I.

Determine the angles ϑ, σ that it is:

$$F(\varphi) + F(\alpha) = F(\sigma), \qquad F(\varphi) - F(\alpha) = F(\vartheta),$$

it will be:

$$\int_0^{\varphi} \frac{k^2 \sin^2 \alpha \cos \alpha \Delta\alpha \sin^2 \varphi d\varphi}{[1 - k^2 \sin^2 \alpha \sin^2 \varphi]\Delta(\varphi)}$$
$$= F(\varphi)E(\alpha) - \frac{1}{2}\int_{\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = F(\varphi)E(\alpha) - \frac{1}{2}\int_0^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} + \frac{1}{2}\int_0^{\vartheta} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

such that the third kind of elliptic integrals depending on three elements, the modulus k, the amplitude φ, the parameter α, are reduced to the first and second kind and the new transcendent:

$$\int_0^{\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

which all depend only on two elements.

50.

Let us put $F(\alpha_2) = 2F(\alpha)$, if $\varphi = \alpha$, it is $\sigma = \alpha_2$, $\vartheta = 0$, in which case we obtain from the propounded theorem:

$$(1.) \qquad \int\limits_0^\alpha \frac{k^2 \sin\alpha\cos\alpha\Delta\alpha\sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)} = F(\alpha)E(\alpha) - \frac{1}{2}\int\limits_0^{\alpha_2} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}.$$

This formula teaches that instead of the new transcendent one can also substitute this one:

$$\int\limits_0^\alpha \frac{\sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)}$$

which is a *definite* integral of the third kind in which the amplitude is equal to the amplitude which therefore also only depends two elements, the modulus k and the quantity which is the parameter and the amplitude at the same time.

Let us put $2F(\mu) = F(\varphi) + F(\alpha) = F(\sigma)$, $2F(\delta) = F(\varphi) - F(\alpha) = F(\vartheta)$, it will be from (1.):

$$\frac{1}{2}\int\limits_0^\sigma \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = F(\mu)E(\mu) - \int\limits_0^\mu \frac{k^2\sin\mu\cos\mu\Delta\mu\sin^2\varphi d\varphi}{[1-k^2\sin^2\mu\sin^2\varphi]\Delta(\varphi)}$$

$$\frac{1}{2}\int\limits_0^\vartheta \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = F(\delta)E(\delta) - \int\limits_0^\delta \frac{k^2\sin\delta\cos\delta\Delta\delta\sin^2\varphi d\varphi}{[1-k^2\sin^2\delta\sin^2\delta]\Delta(\varphi)},$$

having substituted which formulas in theorem propounded in the preceding § we obtain the following

Theorem II.

Determine the angles μ, δ that it is:

$$F(\mu) = \frac{F(\varphi)+F(\alpha)}{2}, \qquad F(\delta) = \frac{F(\varphi)-F(\alpha)}{2},$$

it will be:

$$k^2 \sin\alpha\cos\alpha\Delta\alpha \cdot \int_0^{\varphi} \frac{\sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)} = F(\varphi)E(\alpha) - F(\mu)E(\mu) + F(\delta)E(\delta)$$
$$+ k^2\sin\mu\cos\mu\Delta\mu \cdot \int_0^{\mu} \frac{\sin^2\varphi d\varphi}{[1-k^2\sin^2\mu\sin^2\varphi]\Delta(\varphi)}$$
$$- k^2\sin\delta\cos\delta\Delta\delta \cdot \int_0^{\delta} \frac{\sin^2\varphi d\varphi}{[1-k^2\sin^2\delta\sin^2\varphi]\Delta(\varphi)},$$

by means of which formula the indefinite integrals of the third kind are reduced to definite ones in which the parameter becomes equal to the amplitude, and hence those indefinite integrals that depend on three elements are reduce to other transcendents that consist of only two.

Having interchanged α and φ, ϑ goes over into $-\vartheta$, σ remains unchanged, whence, because moreover it is:

$$\int_{-\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = \int_{+\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

from theorem I:

$$\int_0^{\vartheta} \frac{k^2\sin\alpha\cos\alpha\Delta\alpha\sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)} = F(\varphi)E(\alpha) - \frac{1}{2}\int_{\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}$$

we obtain:

$$\int_0^{\alpha} \frac{k^2\sin\alpha\cos\varphi\Delta\varphi\sin^2\alpha d\alpha}{[1-k^2\sin^2\varphi\sin^2\alpha]\Delta(\alpha)} = F(\alpha)E(\varphi) - \frac{1}{2}\int_{\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}.$$

Hence having done the calculation it arises:

$$(2.) \quad \int_0^{\vartheta} \frac{k^2\sin\alpha\cos\alpha\Delta\alpha\sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)} - \int_0^{\alpha} \frac{k^2\sin\alpha\cos\varphi\Delta\varphi\sin^2\alpha d\alpha}{[1-k^2\sin^2\varphi\sin^2\alpha]\Delta(\alpha)} = F(\varphi)E(\alpha) - F(\alpha)E(\varphi),$$

which formula teaches that *integrals of the third kind can always be reduced to another in which what was the parameter becomes the amplitude and what was the* *amplitude becomes the parameter.*

If in formula (2.) one puts $\varphi = \frac{\pi}{2}$, we obtain:

$$(3.) \qquad \int_0^{\frac{\pi}{2}} \frac{k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)} = F^{\mathrm{I}} E(\alpha) - E^{\mathrm{I}} F(\alpha).$$

Formulas (2.), (3.) agree with those Legendre exhibited in cap. XXIII. pag. 141 (n'), (p').

2.5 The elliptic Integrals of the third kind are expanded into a series. How they are conveniently expressed by means of the new transcendent Θ

51.

From the formula:

$$\sin^2 \mathrm{am} \frac{2K}{\pi}(x + A) - \sin^2 \mathrm{am} \frac{2K}{\pi}(x - A)$$

$$= \frac{4 \sin \mathrm{am} \frac{2KA}{\pi} \cos \mathrm{am} \frac{2KA}{\pi} \Delta \mathrm{am} \frac{2KA}{\pi} \sin \mathrm{am} \frac{2Kx}{\pi} \cos \mathrm{am} \frac{2Kx}{\pi} \Delta \mathrm{am} \frac{2Kx}{\pi}}{\left\{1 - k^2 \sin^2 \mathrm{am} \frac{2KA}{\pi} \sin^2 \mathrm{am} \frac{2Kx}{\pi}\right\}^2}$$

which is known from the elements we find by integrating:

$$(1.) \qquad \frac{2K}{\pi} \int_0^x dx \left\{ \sin^2 \mathrm{am} \frac{2K}{\pi}(x + A) - \sin^2 \mathrm{am} \frac{2K}{\pi}(x - A) \right\}$$

$$= \frac{2 \sin \mathrm{am} \frac{2KA}{\pi} \cos \mathrm{am} \frac{2KA}{\pi} \Delta \mathrm{am} \frac{2KA}{\pi} \sin \mathrm{am} \frac{2Kx}{\pi}}{1 - k^2 \sin^2 \mathrm{am} \frac{2KA}{\pi} \sin^2 \mathrm{am} \frac{2Kx}{\pi}}.$$

In § 41 we already gave the formula:

$$\left(\frac{2kK}{\pi}\right)^2 \sin^2 \mathrm{am} \frac{2Kx}{\pi} = \frac{2K}{\pi}\frac{2K}{\pi} - \frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi} - 4\left\{ \frac{2q\cos 2x}{1 - q^2} + \frac{4q^2 \cos 4x}{1 - q^4} + \frac{6q^3 \cos 6x}{1 - q^6} + \cdots \right\},$$

whence:

$$\left(\frac{2kK}{\pi}\right)^2\left\{\sin^2\operatorname{am}\frac{2K}{\pi}(x+A)-\sin^2\operatorname{am}\frac{2K}{\pi}(x-A)\right\}$$
$$=4\left\{\frac{2q\cos 2(x-A)}{1-q^2}+\frac{4q^2\cos 4(x-A)}{1-q^4}+\frac{6q^3\cos 6(x-A)}{1-q^6}+\cdots\right\}$$
$$-4\left\{\frac{2q\cos 2(x+A)}{1-q^2}+\frac{4q^2\cos 4(x+A)}{1-q^4}+\frac{6q^3\cos 6(x+A)}{1-q^6}+\cdots\right\}$$
$$=8\left\{\frac{2q\sin 2A\sin 2x}{1-q^2}+\frac{4q^2\sin 4A\sin 4x}{1-q^4}+\frac{6q^3\sin 6A\sin 6x}{1-q^6}+\cdots\right\}.$$

Hence, it is from (1.):

$$(2.)\qquad \frac{2K}{\pi}\cdot\frac{2k^2\sin\operatorname{am}\frac{2KA}{\pi}\cos\sin\operatorname{am}\frac{2KA}{\pi}\Delta\sin\operatorname{am}\frac{2KA}{\pi}\sin^2\sin\operatorname{am}\frac{2Kx}{\pi}}{1-k^2\sin^2\sin\operatorname{am}\frac{2KA}{\pi}\sin^2\sin\operatorname{am}\frac{2Kx}{\pi}}$$
$$=\text{const.}+4\left\{\frac{2q\sin 2(x-A)}{1-q^2}+\frac{4q^2\sin 4(x-A)}{1-q^4}+\frac{6q^3\sin 6(x-A)}{1-q^6}+\cdots\right\}$$
$$-4\left\{\frac{2q\sin 2(x+A)}{1-q^2}+\frac{4q^2\sin 4(x+A)}{1-q^4}+\frac{6q^3\sin 6(x+A)}{1-q^6}+\cdots\right\}$$
$$=\text{const.}-8\left\{\frac{2q\sin 2A\cos 2x}{1-q^2}+\frac{4q^2\sin 4A\cos 4x}{1-q^4}+\frac{6q^3\sin 6A\cos 6x}{1-q^6}+\cdots\right\},$$

where the *constant* has to be determined in such a way that the propounded expression vanishes fro $x=0$, whence from § 47 (1.):

$$\text{const.}=8\left\{\frac{q\sin 2A}{1-q^2}+\frac{q^2\sin 4A}{1-q^4}+\frac{q^3\sin 6A}{1-q^6}+\cdots\right\}=2\cdot\frac{2K}{\pi}Z\left(\frac{2KA}{\pi}\right).$$

Having integrated formula (2.) from $x=0$ to $x=\frac{\pi}{2}$, because $\frac{\pi}{2}\cdot$ const. arises, and the other terms vanish, having put $\frac{2KA}{\pi}=a$, $\frac{2Kx}{\pi}=u$, we find the definite integral:

$$\int\limits_0^K\frac{k^2\sin\operatorname{am}a\cos\operatorname{am}a\Delta\operatorname{am}a\sin^2\operatorname{am}u\,du}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u}=K\cdot Z(a),$$

which us the same as (3.) § 50.

In the following we will denote by the character $\Pi(u, a, k)$ or shorter by $\Pi(u, a)$ the integral:

$$\Pi(u,a) = \int_0^u \frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = \int_0^\varphi \frac{k^2 \sin \alpha \cos \alpha \Delta \alpha \sin^2 \varphi d\varphi}{[1 - k^2 \sin^2 \alpha \sin^2 \varphi]\Delta(\varphi)},$$

if $\varphi = \operatorname{am} u$, $\alpha = \operatorname{am} a$. Having put these things and integrated equation (2.) again from $x = 0$ to $x = x$ it arises:

$$\begin{aligned}
(3.) \qquad & \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right) \\
= \frac{2Kx}{\pi} Z\left(\frac{2KA}{\pi}\right) & - \left\{\frac{q \cos 2(x-A)}{1-q^2} + \frac{q^2 \cos 4(x-A)}{2(1-q^4)} + \frac{q^3 \cos 6(x-A)}{3(1-q^6)} + \cdots\right\} \\
& + \frac{q \cos 2(x+A)}{1-q^2} + \frac{q^2 \cos 4(x+A)}{2(1+q^4)} + \frac{q^3 \cos 6(x+A)}{3(1-q^6)} + \cdots \\
= \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right) & - 2\left\{\frac{q \sin 2A \sin 2x}{1-q^2} + \frac{q^2 \sin 4A \sin 4x}{2(1-q^4)} + \frac{q^3 \sin 6A \sin 6x}{3(1-q^6)} + \cdots\right\}
\end{aligned}$$

which is the expansion of the elliptic integral of the third kind we are looking for.

If one recalls the known expansion:

$$-\ln(1 - 2q \cos 2x + q^2) = 2\left\{q \cos 2x + \frac{q^2 \cos 4x}{2} + \frac{q^3 \cos 6x}{3} + \frac{q^4 \cos 8X}{4} + \cdots\right\},$$

we see that formula (3.) having expanded the single denominators $1 - q^2$, $1 - q^4$, $1 - q^6$ etc takes this form:

$$\begin{aligned}
& (4.) \qquad \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right) \\
= \frac{2Kx}{\pi} Z\left(\frac{2KA}{\pi}\right) & + \frac{1}{2} \ln\left\{\frac{(1 - 2q \cos 2(x-A) + q^2)(1 - 2q^3 \cos 2(x-A) + q^6) \cdots}{(1 - 2q \cos 2(x+A) + q^2)(1 - 2q^3 \cos 2(x+A) + q^6) \cdots}\right\}.
\end{aligned}$$

52.

Having integrated the formula (1.) § 47:

$$\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right) = 4\left\{\frac{q\sin 2x}{1-q^2} + \frac{q^2\sin 4x}{1-q^4} + \frac{q^3\sin 6x}{1-q^6} + \cdots\right\}$$

from $x = 0$ to $x = x$ it arises:

$$\frac{2K}{\pi}\int\limits_0^x Z\left(\frac{2Kx}{\pi}\right)dx = -2\left\{\frac{q\cos 2x}{1-q^2} + \frac{q^2\cos 4x}{2(1-q^4)} + \frac{q^3\sin 6x}{3(1-q^6)} + \cdots\right\} + \text{const.}$$

$$= \ln[(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots] + \text{const.}$$

where the *constant* determined in such a way that the integral vanishes for $x = 0$ becomes:

$$2\left\{\frac{q}{1-q^2} + \frac{q^2}{2(1-q^4)} + \frac{q^3}{3(1-q^6)} + \cdots\right\} = -\ln[(1-q)(1-q^3)(1-q^5)\cdots]^2,$$

and hence:

$$(1.)\qquad \frac{2K}{\pi}\int\limits_0^x Z\left(\frac{2Kx}{\pi}\right)dx = \ln\left\{\frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}\right\}.$$

In the following, we will denote by the character $\Theta(u)$ the expression:

$$\Theta(u) = \Theta(0)e^{\int\limits_0^u Z(u)du},$$

where $\Theta(0)$ denotes the constant that we leave undetermined for the moment; we will obtain a convenient determination of it below; it will be from (1.):

$$(2.)\quad \frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

whence formula (4.) § 51 goes over into this one:

$$\Pi\left(\frac{2Kx}{\pi},\frac{2KA}{\pi}\right) = \frac{2Kx}{\pi}Z\left(\frac{2KA}{\pi}\right) + \frac{1}{2}\ln\frac{\Theta\left(\frac{2K}{\pi}(x-A)\right)}{\Theta\left(\frac{2K}{\pi}(x+A)\right)},$$

or having put $\frac{2Kx}{\pi} = u$, $\frac{2KA}{\pi} = a$ again:

$$(3.) \qquad \Pi(u,a) = uZ(a) + \frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)} = u\frac{\Theta'(a)}{\Theta(a)} + \frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)},$$

if it is put: $\frac{d\Theta(u)}{du} = \Theta'(u)$. This is a comfortable expression for the elliptic integral Π be the new transcendent Θ.

It easily becomes clear that $\Theta(-u) = \Theta(u)$, whence having interchanged a and u it arises from (3.):

$$\Pi(a,u) = aZ(u) + \frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)},$$

having subtracted which from (3.) it is:

$$(4.) \qquad \Pi(u,a) - \Pi(a,u) = uZ(a) - aZ(u),$$

which is the same a formula (2.) § 50. Hence, having put $\Pi(K,a) = \Pi^{\mathrm{I}}(a)$, since $\Pi(a,K)$, $Z(K)$, it is:

$$\Pi^{\mathrm{I}}(a) = KZ(a),$$

which is Legendre's formula which exhibited above (3.) § 50.

Having put $u = a$, it is from (3.):

$$(5.) \qquad \Pi(a,a) = aZ(a) + \frac{1}{2}\ln\frac{\Theta(0)}{\Theta(2a)} = aZ(a) - \ln\frac{\Theta(2a)}{\Theta(0)}.$$

Therefore, we see that the new transcendent can defined either defined by the integral $\int \frac{E(\varphi)d\varphi}{\Delta(\varphi)}$ by means of the formula:

$$(6.) \qquad \frac{\Theta(u)}{\Theta(0)} = e^{\int\limits_0^u Z(u)du} = e^{\int\limits_0^\varphi \frac{F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}\Delta(\varphi)}\cdot d\varphi},$$

or by a definite integral of the third kind by means of the formula:

$$(7.) \qquad \frac{\Theta(2a)}{\Theta(0)} = e^{2aZ(a) - 2\Pi(a,a)}.$$

From formula (5.) we obtain:

$$\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)} = \frac{u-a}{2}Z\left(\frac{u-a}{2}\right) - \Pi\left(\frac{u-a}{2},\frac{u-a}{2}\right)$$
$$-\frac{u+a}{2}Z\left(\frac{u+a}{2}\right) + \Pi\left(\frac{u+a}{2},\frac{u+a}{2}\right),$$

whence (3.) goes over into this formula:

$$\text{(8.)}\qquad \Pi(u,a) = uZ(a) + \frac{u-a}{2}Z\left(\frac{u-a}{2}\right) - \frac{u+a}{2}Z\left(\frac{u+a}{2}\right)$$
$$-\Pi\left(\frac{u-a}{2},\frac{u-a}{2}\right) + \Pi\left(\frac{u+a}{2},\frac{u+a}{2}\right)$$

which is the formula for the reduction of an indefinite integral of the third kind to definite ones and agrees with Theorem II. § 50.

Corollary

Above we already deduced suitable algorithms for the computation from the found expansions; instead of showing new things, but to better understand the nature of the things said let us treat the same on the invention of expansion of the function:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = e^{\int\limits_0^{\varphi}\frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}\Delta(\varphi)}d\varphi}$$
$$= \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}.$$

For this aim, we want to mention the following things in advance.

Put the infinite product:

$$T = \left(\frac{1-q}{1+q}\right)\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{2}}\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{4}}\left(\frac{1-q^8}{1+q^8}\right)^{\frac{1}{8}}\cdots,$$

if one substitutes the following again and again:

$$1-q^2 = (1-q)(1+q),\quad 1-q^4 = (1-q^2)(1+q^2),\quad 1-q^8 = (1-q^4)(1+q^4),\cdots,$$

it arises:

$$\begin{aligned}
T &= (1-q)\cdot\left(\frac{1-q}{1+q}\right)^{\frac{1}{2}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{8}}\cdot\left(\frac{1-q^8}{1+q^8}\right)^{\frac{1}{16}}\cdots\\
&= (1-q)\cdot\quad(1-q)^{\frac{1}{2}}\quad\cdot\quad\left(\frac{1-q}{1+q}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{8}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{16}}\cdots\\
&= (1-q)\cdot\quad(1-q)^{\frac{1}{2}}\quad\cdot\quad(1-q)^{\frac{1}{4}}\quad\cdot\quad\left(\frac{1-q}{1+q}\right)^{\frac{1}{8}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{16}}\cdots\\
&\cdots\cdots
\end{aligned}$$

whence we see that it will be:

$$(1.)\qquad T=(1-q)(1-q)^{\frac{1}{2}}(1-q)^{\frac{1}{4}}(1-q)^{\frac{1}{8}}(1-q)^{\frac{1}{16}}\cdots=(1-q)^2$$

Or also, because it is:

$$\begin{aligned}
T &= \left(\frac{1-q}{1+q}\right)\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{2}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^8}{1+q^8}\right)^{\frac{1}{8}}\cdots\\
&= \quad(1-q)\quad\cdot\quad\left(\frac{1-q}{1+q}\right)^{\frac{1}{2}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{8}}\cdots,
\end{aligned}$$

it is $T=(1-q)\sqrt{T}$, whence $T=(1-q)^2$.

Therefore, it is

$$(2.)\qquad 1-q=\left(\frac{1-q}{1+q}\right)^{\frac{1}{2}}\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{4}}\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{8}}\cdots,$$

in which formula we want successively put q, q^3, q^5, q^7 etc. instead of q. Recalling and using the formula exhibited above:

$$\sqrt[4]{k'}=\left(\frac{1-q}{1+q}\right)\left(\frac{1-q^3}{1+q^3}\right)\left(\frac{1-q^5}{1+q^5}\right)\left(\frac{1-q^7}{1+q^7}\right)\cdots,$$

it arises:

$$(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots=[k']^{\frac{1}{8}}[k^{(2)\prime}]^{\frac{1}{16}}[k^{(4)\prime}]^{\frac{1}{32}}\cdots,$$

if we denote, as above, by $k^{(r)\prime}$ the quantity which depends the same way on q^r as k' on q, or the complement of the modulus found by first transformation of r-th order.

Further, we find § 36:

$$\{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots\}^6 = \frac{2\sqrt[4]{q}k'}{\sqrt{k}},$$

whence now:

$$(3.) \qquad q = e^{\frac{-\pi K'}{K}} = \frac{kk}{16k'}[k^{(2)\prime}]^{\frac{3}{2}}[k^{(4)\prime}]^{\frac{3}{4}}[k^{(8)\prime}]^{\frac{3}{8}}\cdots$$

Having put $m = 1$, $n = k'$; $\frac{m+n}{2} = m'$, $\sqrt{mn} = n'$; $\frac{m'+n'}{2} = m''$, $\sqrt{m'n'} = n''$, etc., its is known that $k^{(2)\prime} = \frac{m'}{n'}$, $k^{(4)\prime} = \frac{m''}{n''}$, $k^{(8)\prime} = \frac{m'''}{n'''}$, etc., whence:

$$(4.) \qquad q = \frac{mm-nn}{16mn}\cdot\left\{\left(\frac{n'}{m'}\right)^{\frac{1}{2}}\left(\frac{n''}{m''}\right)^{\frac{1}{4}}\left(\frac{n'''}{m'''}\right)^{\frac{1}{8}}\cdots\right\}^3.$$

Hence, it also follows, because $\mu = \frac{\pi}{2K}$ denotes the common limit, to which the quantities $m^{(p)}$, $n^{(p)}$ converge:

$$(5.) \qquad K' = \frac{1}{2\mu}\left\{\ln\frac{16mn}{mm-nn} + \frac{3}{2}\ln\frac{m'}{n'} + \frac{3}{4}\ln\frac{m''}{n''} + \frac{3}{8}\ln\frac{m'''}{n'''} + \cdots\right\}$$

which formulas allow a very fast calculation. (5.) tells us, how from the same series of quantities which one needs to have calculated in order to find the value of the function K, the value of K' also immediately arises.

Let us transform formula (3.). It is, as it is known:

$$k' = \frac{1-k^{(2)}}{1+k^{(2)}}; \quad k = \frac{2\sqrt{k^{(2)}}}{1+k^{(2)}}, \quad \text{whence} \quad \frac{kk}{k'} = \frac{4k^{(2)}}{k^{(2)\prime}k^{(2)\prime}}.$$

Hence, we obtain, if we substitute $k^{(2)}$ instead of k again and again and take the square root:

$$\begin{aligned}
\frac{kk}{16k'} \quad &\cdot \left\{k^{(2)\prime}\right\}^{\frac{3}{2}} = \left\{\frac{k^{(2)}k^{(2)}}{16k^{(2)\prime}}\right\}^{\frac{1}{2}} \\
\left\{\frac{k^{(2)}k^{(2)}}{16k^{(2)\prime}}\right\}^{\frac{1}{2}} &\cdot \left\{k^{(4)\prime}\right\}^{\frac{3}{4}} = \left\{\frac{k^{(4)}k^{(4)}}{16k^{(4)\prime}}\right\}^{\frac{1}{4}} \\
\left\{\frac{k^{(4)}k^{(4)}}{16k^{(4)\prime}}\right\}^{\frac{1}{4}} &\cdot \left\{k^{(8)\prime}\right\}^{\frac{3}{8}} = \left\{\frac{k^{(8)}k^{(8)}}{16k^{(8)\prime}}\right\}^{\frac{1}{8}} \\
\cdots,
\end{aligned}$$

whence having put $r = 2^p$:

$$\frac{kk}{16k'} = \left\{k^{(2)\prime}\right\}^{\frac{3}{2}} \left\{k^{(4)\prime}\right\}^{\frac{3}{4}} \left\{k^{(8)\prime}\right\}^{\frac{3}{8}} \cdots \left\{k^{(r)\prime}\right\}^{\frac{3}{r}} = \left\{\frac{k^{(r)}k^{(r)}}{16k^{(r)\prime}}\right\}^{\frac{1}{r}}.$$

Hence, we see that from formula (3.) $q = e^{\frac{-\pi K'}{K}}$ will be the limit of the expression $\left\{\frac{k^{(r)}k^{(r)}}{16k^{(r)\prime}}\right\}^{\frac{1}{r}}$, as p or r increases to infinity which is the theorem found by Legendre.

And it is immediately clear from the formula exhibited by us:

$$k = 4\sqrt{q}\left\{\frac{(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots}{(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots}\right\}^4$$

that having neglected the quantities of order q^r it will be:

$$q = \sqrt[r]{\frac{k^{(r)}k^{(r)}}{16}},$$

which agrees with the mentioned theorem.

Now, let us put in our formula:

$$1 - q = \left\{\frac{1-q}{1+q}\right\}^{\frac{1}{2}} \left\{\frac{1-q^2}{1+q^2}\right\}^{\frac{1}{4}} \left\{\frac{1-q^4}{1+q^4}\right\}^{\frac{1}{8}} \cdots$$

instead of q successively put the two series of quantities:

$$
\begin{array}{llll}
qe^{2ix}, & q^3e^{2ix}, & q^5e^{2ix}, & q^7e^{2ix}, \cdots \\
qe^{-2ix}, & q^3e^{-2ix}, & q^5e^{-2ix}, & q^7e^{-2ix}, \cdots
\end{array}
$$

and multiply the infinitely many terms. Recall the formula of § 36:

$$\Delta \operatorname{am} \frac{2Kx}{\pi} = \sqrt{k'} \frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

and let us denote by $\Delta^{(r)}$ the expression

$$\Delta \operatorname{am}\left(\frac{2rK^{(r)}x}{\pi}, k^{(r)}\right) = \sqrt{k^{(r)\prime}} \frac{(1+2q^r\cos 2x+q^{2r})(1+2q^{3r}\cos 2x+q^{6r})(1+2q^{5r}\cos 2x+q^{10r})\cdots}{(1-2q^r\cos 2x+q^{2r})(1-2q^{3r}\cos 2x+q^{6r})(1-2q^{5r}\cos 2x+q^{10r})\cdots},$$

it arises:

$$\frac{1}{\Delta^{(1)\frac{1}{2}}\Delta^{(2)\frac{1}{4}}\Delta^{(4)\frac{1}{8}}\Delta^{(8)\frac{1}{16}}\cdots} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

We determined the constant factor we added, $\frac{1}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}$, from the thing found above or from that both expression for $x=0$ become equal to unity. But now we find:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

whence:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{1}{\Delta^{(1)\frac{1}{2}}\Delta^{(2)\frac{1}{4}}\Delta^{(4)\frac{1}{8}}\Delta^{(8)\frac{1}{16}}\cdots}$$

Hence having put $\frac{2kX}{\pi} = u$, $\operatorname{am} u = \varphi$ and having recalled the formulas that Legendre propounded on the transformation of the second order we obtain the following theorem which yields a fast way for the calculation of the function Θ.

Theorem

Put $\operatorname{am} u = \varphi$, $m = 1$, $n = k'$, $\Delta(\varphi) = \sqrt{mm\cos^2\varphi + nn\sin^2\varphi} = \Delta$ and calculate the series of quantities

$$\begin{array}{llll}
m' = \dfrac{m+n}{2}, & m'' = \dfrac{m'+n'}{2}, & m''' = \dfrac{m''+n''}{2}, & \cdots \\
n' = \sqrt{mn}, & n'' = \sqrt{m'n'}, & n''' = \sqrt{m''n''}, & \cdots \\
\Delta' = \dfrac{\Delta\Delta + n'n'}{2\Delta}, & \Delta'' = \dfrac{\Delta'\Delta' + n''n''}{2\Delta'}, & \Delta''' = \dfrac{\Delta''\Delta'' + n'''n'''}{2\Delta''}, & \cdots
\end{array}$$

it will be:

$$\frac{\Theta(u)}{\Theta(0)} = e^{\int\limits_0^{\varphi} \frac{F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}\Delta(\varphi)} d\varphi} = \left\{\frac{m}{\Delta}\right\}^{\frac{1}{2}} \cdot \left\{\frac{m'}{\Delta'}\right\}^{\frac{1}{4}} \cdot \left\{\frac{m''}{\Delta''}\right\}^{\frac{1}{8}} \cdot \left\{\frac{m'''}{\Delta'''}\right\}^{\frac{1}{16}} \cdots$$

We put aside the task to demonstrate this theorem and the consideration of expansions by means of known and finite formulas because both are easily done.

2.6 On the Addition of Arguments both of the parameter and the amplitude in the elliptic integrals of the first kind

53.

The fundamental formula in the analysis of the function Θ which we will use very frequently in the following we obtain from the following consideration. For, because it was put:

$$\Pi(u,a) = \int\limits_0^u \frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 am \operatorname{am} u},$$

it is:

$$\frac{d\Pi(u,a)}{du} = \frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 am \operatorname{am} u}.$$

Having integrated this formula with respect to a from $a = 0$ to $a = a$, it arises:

$$(1.)\qquad \int_0^a da \frac{d\Pi(u,a)}{da} = -\frac{1}{2}\ln(1 - k^2 \sin^2 \operatorname{am} \sin^2 \operatorname{am} u).$$

But from (3.) § 52 it is:

$$(2.)\qquad \frac{d\Pi(u,a)}{du} = Z(a) + \frac{1}{2}\frac{\Theta'(u-a)}{\Theta(u-a)} - \frac{1}{2}\frac{\Theta'(u+a)}{\Theta(u+a)},$$

whence:

$$\int_0^a da \frac{d\Pi(u,a)}{du} = \ln\frac{\Theta(a)}{\Theta(0)} - \frac{1}{2}\ln\Theta(u-a) - \frac{1}{2}\ln\Theta(u+a) + \ln\Theta(u),$$

having substituted which going from logarithms to ordinary logarithms one obtains from (1.):

$$(3.)\qquad \Theta(u+a)\Theta(u-a) = \left\{\frac{\Theta(u)\Theta(a)}{\Theta(0)}\right\}^2 (1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u).$$

We can represent formula (2.) this way:

$$\frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = Z(u) + \frac{1}{2}Z(u-a) - \frac{1}{2}Z(u+a),$$

whence having commuted a and u:

$$\frac{k^2 \sin \operatorname{am} u \cos \operatorname{am} u \Delta \operatorname{am} u \sin^2 \operatorname{am} a}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = Z(u) - \frac{1}{2}Z(u-a) - \frac{1}{2}Z(u+a),$$

having added which formulas it arises:

$$(4.)\qquad Z(u) + Z(a) - Z(u+a) = k^2 \sin \operatorname{am} u \sin \operatorname{am} a \sin \operatorname{am}(u+a),$$

which is for the addition of the function Z and agrees with formula (3.) § 49:

$$E(\varphi) + E(\alpha) - E(\sigma) = k^2 \sin\varphi \sin\alpha \sin\sigma.$$

Having put $a = K$, because it easily becomes clear that $Z(K) = \frac{F^{\mathrm{I}}E^{\mathrm{I}} - E^{\mathrm{I}}F^{\mathrm{I}}}{F^{\mathrm{I}}} = 0$, it arises from (4.):

$$(5.) \qquad Z(u) - Z(u + K) = k^2 \sin \operatorname{am} u \sin \operatorname{coam} u,$$

which in § 47 we derived from the expansion of Z. Having put $-u$ instead of u, $K - u = v$, we obtain from formula (5.):

$$(6.) \qquad Z(u) + Z(v) = k^2 \sin \operatorname{am} u \sin \operatorname{am} v.$$

Having put $u = v = \frac{K}{2}$ it is $2Z\left(\frac{K}{2}\right) = 1 - k'$.

Let us integrate formula (5.) from $u = 0$ to $u = u$. Because it is $\int\limits_0^u Z(u)du = \ln \frac{\Theta(u)}{\Theta(0)}$, it arises:

$$\ln \frac{\Theta(u)}{\Theta(0)} - \ln \frac{\Theta(u + K)}{\Theta(K)} = -\ln \Delta \operatorname{am} u$$

or:

$$(7.) \qquad \frac{\Theta(0)}{\Theta(K)} \cdot \frac{\Theta(u + K)}{\Theta(u)} = \Delta \operatorname{am} u.$$

Having put $u = -K$ we find from (7.) the value of:

$$(8.) \qquad \frac{\Theta(K)}{\Theta(0)} = \frac{1}{\sqrt{k'}},$$

whence (7.) take the form:

$$(9.) \qquad \frac{\Theta(u + K)}{\Theta(u)} = \frac{\Delta \operatorname{am} u}{\sqrt{k'}}.$$

We easily confirm formula (9.) from the found expansion:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1 - 2q\cos 2x + q^2)(1 - 2q^3\cos 2x + q^6)(1 - 2q^5\cos 2x + q^{10})\cdots}{[(1 - q)(1 - q^3)(1 - q^5)\cdots]^2}.$$

For, it is having changed x to $x + \frac{\pi}{2}$:

$$\frac{\Theta\left(\frac{2Kx}{\pi} + K\right)}{\Theta(0)} = \frac{(1 + 2q\cos 2x + q^2)(1 + 2q^3\cos 2x + q^6)(1 + 2q^5\cos 2x + q^{10})\cdots}{[(1 - q)(1 - q^3)(1 - q^5)\cdots]^2},$$

whence:

$$\frac{\Theta\left(\frac{2Kx}{\pi}+K\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}=\frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

which expression we in § 35 found to be $=\frac{\Delta\operatorname{am}\frac{2Kx}{\pi}}{\sqrt{k'}}$ as is has to be.

From formula (9.) the expression $\Pi(u+K,a)$, $\Pi(u,a+K)$ we immediately are led back to $\Pi(u,a)$. For, it is:

$$\begin{aligned}
(10.)\qquad \Pi(u+K,a) &= (u+K)Z(a)+\frac{1}{2}\ln\frac{\Theta(u+K-a)}{\Theta(u+K+a)}\\
&= (u+K)Z(a)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}\\
&= \Pi(u,a)+KZ(a)+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}\\
(11.)\qquad \Pi(u,a+K) &= uZ(a+K)+\frac{1}{2}\ln\frac{\Theta(u-a-K)}{\Theta(u+a+K)}\\
&= uZ(a)-k^2\sin\operatorname{am}a\sin\operatorname{coam}a\cdot u+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}\\
&= \Pi(u,a)-k^2\sin\operatorname{am}a\sin\operatorname{coam}a\cdot u+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}.
\end{aligned}$$

54.

From the fundamental formula, by means of which the function Π is defined by Z, Θ:

$$\text{I.}\qquad \Pi(u,a)=uZ(a)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)},$$

and having recalled the following fundamental ones in the analysis of the functions Z, Θ:

$$\text{II.}\qquad Z(u)+Z(a)-Z(u+a)=k^2\sin\operatorname{am}a\sin\operatorname{am}u\operatorname{am}(u+a)$$

$$\text{III.}\qquad \Theta(u+a)\Theta(u-a)=\left\{\frac{\Theta(u)\Theta(a)}{\Theta(0)}\right\}^2(1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u),$$

one now easily obtains formulas both for expressing $\Pi(u+v,a)$ by means of $\Pi(u,a)$, $\Pi(v,a)$, which we will call the theorem on *the addition of the argument of the amplitude*, and for expressing $\Pi(u,a+b)$ by means of $\Pi(u,a)$, $\Pi(u,b)$, which we will call the theoremthe addition of the argument of the parameter. For this purpose, we add the following things.

From the formulas:

$$\begin{aligned}
\Pi(u,a) &= uZ(a) &&+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}\\
\Pi(v,a) &= vZ(a) &&+\frac{1}{2}\ln\frac{\Theta(v-a)}{\Theta(v+a)}\\
\Pi(u+v,a) &= (u+v)Z(a) &&+\frac{1}{2}\ln\frac{\Theta(u+v-a)}{.}\Theta(u+v+a)
\end{aligned}$$

it follows:

$$\text{(1.)}\qquad \Pi(u,a)+\Pi(v,a)-\Pi(u+v,a)=\frac{1}{2}\ln\frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}.$$

The expression contained in the logarithm:

$$\frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}$$

can be reduced to elliptic function by means of fundamental theorem (III.) in two ways. For, first it is from the theorem:

$$\begin{aligned}
\Theta(u-a)\Theta(v-a) &= \left\{\frac{\Theta\left(\frac{u-v}{2}\right)\Theta\left(\frac{u+v}{2}-a\right)}{\Theta(0)}\right\}^2\cdot\left(1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right)\\
\Theta(u+a)\Theta(v+a) &= \left\{\frac{\Theta\left(\frac{u-v}{2}\right)\Theta\left(\frac{u+v}{2}+a\right)}{\Theta(0)}\right\}^2\cdot\left(1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right)\\
\Theta(u+v-a)\Theta(a) &= \left\{\frac{\Theta\left(\frac{u+v}{2}\right)\Theta\left(\frac{u+v}{2}-a\right)}{\Theta(0)}\right\}^2\cdot\left(1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right)\\
\Theta(u+v+a)\Theta(a) &= \left\{\frac{\Theta\left(\frac{u+v}{2}\right)\Theta\left(\frac{u+v}{2}+a\right)}{\Theta(0)}\right\}^2\cdot\left(1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right)
\end{aligned}$$

after having multiplied the first and the fourth and divided by the second and the third of which formulas it arises:

$$(2.)\qquad \frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}$$

$$= \frac{\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}+a\right)\right\}}{\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}+a\right)\right\}\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}}.$$

So also, which is the other way, where fundamental theorem (III.) is represented as this:

$$\left\{\frac{\Theta(u)\theta(v)}{\Theta(0)}\right\}^2 = \frac{\Theta(u+v)\Theta(u-v)}{1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v},$$

it is:

$$\begin{aligned}
\left\{\frac{\Theta(u-a)\Theta(v-a)}{\Theta(0)}\right\}^2 &= \frac{\Theta(u-v)\Theta(u+v-2a)}{1-k^2\sin^2\mathrm{am}(u-a)\sin^2\mathrm{am}(v-a)}\\
\left\{\frac{\Theta(u+a)\Theta(v+a)}{\Theta(0)}\right\}^2 &= \frac{\Theta(u-v)\Theta(u+v+2a)}{1-k^2\sin^2\mathrm{am}(u+a)\sin^2\mathrm{am}(v+a)}\\
\left\{\frac{\Theta(a)\Theta(u+v-a)}{\Theta(0)}\right\}^2 &= \frac{\Theta(u+v)\Theta(u+v-2a)}{1-k^2\sin^2\mathrm{am}(a)\sin^2\mathrm{am}(u+v-a)}\\
\left\{\frac{\Theta(a)\Theta(u+v+a)}{\Theta(0)}\right\}^2 &= \frac{\Theta(u+v)\Theta(u+v+2a)}{1-k^2\sin^2\mathrm{am}(a)\sin^2\mathrm{am}(u+v+a)},
\end{aligned}$$

after again having multiplied the first and the fourth by each other and divided the divided by the second and the third of which formulas and taken the square root it arises:

$$(3.)\qquad \frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}$$

$$= \sqrt{\frac{[1-k^2\sin^2\mathrm{am}(u+a)\sin^2\mathrm{am}(v+a)][1-k^2\sin^2\mathrm{am}\,a\sin^2\mathrm{am}(u+v-a)]}{[1-k^2\sin^2\mathrm{am}(u-a)\sin^2\mathrm{am}(v-a)][1-k^2\sin^2\mathrm{am}\,a\sin^2\mathrm{am}(u+v+a)]}}.$$

To see from the elements itself how the one of the expressions (2.),(3.) can be transformed into the other, I mention the following.

If in the formula, already frequently used,

$$\sin\operatorname{am}(u+v)\sin\operatorname{am}(u-v)=\frac{\sin^2\operatorname{am}u-\sin^2\operatorname{am}v}{1-k^2\operatorname{am}u\sin^2\operatorname{am}v}$$

instead of u, v one puts $u+v$, $u-v$ respectively, it arises:

$$\sin\operatorname{am}2u\sin\operatorname{am}2v=\frac{\sin^2\operatorname{am}(u+v)-\sin^2\operatorname{am}(u-v)}{1-k^2\sin^2\operatorname{am}(u+v)\sin^2\operatorname{am}(u-v)}.$$

Further, we gave the formula:

$$\sin^2\operatorname{am}(u+v)-\sin^2\operatorname{am}(u-v)=\frac{4\sin\operatorname{am}u\cos\operatorname{am}u\Delta\operatorname{am}u\sin\operatorname{am}v\cos\operatorname{am}v\Delta\operatorname{am}v}{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]^2},$$

whence having done the multiplication we obtain:

$$(4.)\quad 1-k^2\sin^2\operatorname{am}(u+v)\sin^2\operatorname{am}(u-v)=\frac{4\sin\operatorname{am}u\cos\operatorname{am}u\Delta\operatorname{am}u\sin\operatorname{am}v\cos\operatorname{am}v\Delta\operatorname{am}v}{\sin\operatorname{am}2u\sin\operatorname{am}2v[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]^2}$$
$$=\frac{[1-k^2\sin^4\operatorname{am}u][1-k^2\sin^4\operatorname{am}v]}{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]^2}$$

by means of which formula the one of the formulas (2.), (3.) can now easily be deduced from the other.

From formula (4.) one can also deduce this more general one:

$$(5.)\qquad \frac{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v][1-k^2\sin^2\operatorname{am}u'\sin^2\operatorname{am}v']}{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}u'][1-k^2\sin^2\operatorname{am}v\sin^2\operatorname{am}v']}$$
$$=\sqrt{\frac{[1-k^2\sin^2\operatorname{am}(u+u')\sin^2\operatorname{am}(u-u')][1-k^2\sin^2\operatorname{am}(v+v')\sin^2\operatorname{am}(v-v')]}{[1-k^2\sin^2\operatorname{am}(u+v)\sin^2\operatorname{am}(u-v)][1-k^2\sin^2\operatorname{am}(u'+v')\sin^2\operatorname{am}(u'-v')]}}.$$

But Legendre when he treated the addition of the argument of the amplitude (cap. XVI. *Comparison des fonctions elliptiques de la troisieme espece*) exhibited the quantity which in found within the logarithm in this form:

$$\frac{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)}{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v+a)},$$

which is not obvious at first sight how it coincides with the expressions (2.) and (3.) we found. The rather intricate transformation is done this way.

From the elementary formula which we have already used very frequently it is:

$$\begin{aligned}
\sin\operatorname{am} u \sin\operatorname{am} v &= \frac{\sin^2\operatorname{am}\left(\frac{u+v}{2}\right) - \sin^2\left(\frac{u-v}{2}\right)}{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)} \\
\sin\operatorname{am} a \sin\operatorname{am}(u+v-a) &= \frac{\sin^2\operatorname{am}\left(\frac{u+v}{2}\right) - \sin^2\left(\frac{u+v}{2} - a\right)}{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)};
\end{aligned}$$

having multiplied them by each other, it arises:

$$\begin{aligned}
&\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\right\}\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\} \\
&\qquad \times \left\{1 - k^2\sin\operatorname{am} u \sin\operatorname{am} v \sin\operatorname{am}(u+v-a)\right\} \\
&= \left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\right\}\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\} \\
&-k^2\left\{\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am} - \left(\frac{u-v}{2}\right)\right\}\left\{\sin^2\operatorname{am}\left(\frac{u+v}{2}\right) - \sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\}
\end{aligned}$$

The one side of the equation by canceling them terms

$$\begin{aligned}
&- k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\left\{\sin^2\operatorname{am}\left(\frac{u-v}{2}\right) + \sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\} \\
&+ k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\left\{\sin^2\operatorname{am}\left(\frac{u-v}{2}\right) + \sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\}
\end{aligned}$$

it is:

$$\begin{aligned}
&1 + k^4\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right) \\
&-k^2\sin^4\operatorname{am}\left(\frac{u+v}{2}\right) - k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right) \\
&= \left\{1 - k^2\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)\right\}\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\},
\end{aligned}$$

whence it finally arises:

$$(6.)\qquad \frac{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)}{1-k^2\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)}\left\{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)\right\}$$
$$=\frac{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)}{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)},$$

whence after a division:

$$(7.)\qquad \frac{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)}{1+k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v+a)}$$
$$=\frac{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)}{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)}\cdot\frac{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)}{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)},$$

which is transformation we looked for of the expression propounded by Legendre into expression (2.).

Formula (6.) having put u, a, v instead of $\frac{u-v}{2}$, $\frac{u+v}{2}$, $\frac{u+v}{2}-a$ can also be represented this way:

$$(8.)\qquad 1-k^2\sin\operatorname{am}(a+u)\sin\operatorname{am}(a-u)\sin\operatorname{am}(a+v)\sin\operatorname{am}(a-v)$$
$$=\frac{[1-k^2\sin^4\operatorname{am}a][1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]}{[1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u][1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}v]},$$

whence formula (4.) follows as a special case having put $u=v$.

55.

From the formulas (1.), (2.), (3.), (7.) of the preceding § it follows:

$$(1.)\qquad \Pi(u,a)+\Pi(v,a)-\Pi(u+v,a)$$
$$=\frac{1}{2}\ln\frac{\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right\}\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right\}}{\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right\}\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right\}}$$
$$=\frac{1}{4}\ln\frac{[1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+a)][1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+v-a)]}{[1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-a)][1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+v+a)]}$$
$$=\frac{1}{2}\ln\frac{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)}{1+k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v+a)},$$

which is the theorem on the addition of the argument of the *amplitude*. Further, by the same method one can investigate the other on the addition of the argument of the *parameter*, but by means of the theorem on the reduction of the parameter of the amplitude, which formula (4.) § 52 gave us:

$$\text{(IV.)} \qquad \Pi(u,a) - \Pi(a,u) = uZ(a) - aZ(u),$$

from which formula (1.) immediately follows. For, it is from (IV.):

$$\begin{aligned}
\Pi(a,u) \quad &- \Pi(u,a) \quad &&= aZ(u) \quad &&- uZ(a) \\
\Pi(b,u) \quad &- \Pi(u,b) \quad &&= bZ(u) \quad &&- uZ(b) \\
\Pi(a+b,u) &- \Pi(u,a+b) &&= (a+b)Z(u) &&- uZ(a+b),
\end{aligned}$$

whence:

$$\begin{aligned}
&\Pi(u,a) + \Pi(u,b) - \Pi(u,a+b) \\
=&\Pi(a,u) + \Pi(b,u) - \Pi(a+b,u) + u[Z(a) + Z(b) - Z(a+b)],
\end{aligned}$$

or because it is from (1.):

$$\Pi(a,u) + \Pi(b,u) - \Pi(a+b,u) = \frac{1}{2} \ln \frac{1 - k^2 \sin\operatorname{am} u \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b-u)}{1 + k^2 \sin\operatorname{am} u \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b+u)},$$

further, from (II.):

$$Z(a) + Z(b) - Z(a+b) = k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b),$$

it is:

$$\begin{aligned}
&\text{(2.)} \qquad \Pi(u,a) + \Pi(u,b) - \Pi(u,a+b) \\
&= k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b) \cdot u + \frac{1}{2} \ln \frac{1 - k^2 \sin\operatorname{am} u \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b-u)}{1 + k^2 \sin\operatorname{am} u \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b+u)},
\end{aligned}$$

which is the looked-for theorem on the addition of the argument of the *parameter*.

We find other also remarkable formulas by the following consideration. For, it is from theorem (III.):

$$\left\{\frac{\Theta(u-a)\Theta(v-b)}{\Theta(0)}\right\}^2 = \frac{\Theta(u+v-a-b)\Theta(u-v-a+b)}{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-b)}$$
$$\left\{\frac{\Theta(u+a)\Theta(v+b)}{\Theta(0)}\right\}^2 = \frac{\Theta(u+v+a+b)\Theta(u-v+a-b)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+b)}.$$

Now, it will be from theorem (I.):

$$\Pi(u,a)+\Pi(v,b) = uZ/a)+vZ(b)+\frac{1}{2}\ln\frac{\Theta(u-a)\Theta(v-b)}{\Theta(u+a)\Theta(v+b)}$$
$$\Pi(u+v,a+b)+\Pi(u-v,a-b)$$
$$=(u+v)Z(a+b)+(u-v)Z(a-b)+\frac{1}{2}\ln\frac{\Theta(u+v-a-b)\Theta(u-v-a+b)}{\Theta(u+v+a+b)\Theta(u-v+a-b)},$$

whence:

$$(3.)\qquad \Pi(u+v,a+b)+\Pi(u-v,a-b)-2\Pi(u,a)-2\Pi(v,b)$$
$$=(u+v)Z(a+b)+(u-v)Z(a-b)-2uZ(a)-2vZ(b)+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-b)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+b)},$$

or because it is:

$$Z(a)+Z(b)-Z(a+b) = +k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(a+b)$$
$$Z(a)-Z(b)-Z(a-b) = -k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(a-b),$$

it arises:

$$(4.)\qquad \Pi(u+v,a+b)+\Pi(u-v,a-b)-2\Pi(u,a)-2\Pi(v,b)$$
$$=-k^2\sin\operatorname{am}a\sin\operatorname{am}b[\sin\operatorname{am}(a+b)\cdot(u+v)-\sin\operatorname{am}(a-b)\cdot(u-v)]$$
$$+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-b)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+b)}.$$

Having interchanged u and v we obtain:

$$(5.)\qquad \Pi(u+v,a+b)-\Pi(u-v,a-b)-2\Pi(v,a)-2\Pi(u,b)$$
$$=-k^2\sin\operatorname{am}a\sin\operatorname{am}b[\sin\operatorname{am}(a+b)\cdot(u+v)+\sin\operatorname{am}(a-b)\cdot(u-v)]$$
$$+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}(v-a)\sin^2\operatorname{am}(u-b)}{1-k^2\sin^2\operatorname{am}(v+a)\sin^2\operatorname{am}(u+b)}.$$

Having added (4.) and (5.) we obtain:

$$(6.)\qquad \Pi(u+v,a+b)-\Pi(u-v,a-b)-2\Pi(v,a)-2\Pi(u,b)$$
$$=-k^2\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(a+b)\cdot(u+v)$$
$$+\frac{1}{4}\ln\left\{\frac{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-b)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+b)}\cdot\frac{1-k^2\sin^2\operatorname{am}(v-a)\sin^2\operatorname{am}(u-b)}{1-k^2\sin^2\operatorname{am}(v+a)\sin^2\operatorname{am}(u+b)}\right\}.$$

Having put $v=0$ it arises from (4.), (5.):

$$(7.)\qquad \Pi(u,a+b)+\Pi(u,a-b)-2\Pi(u,a)$$
$$-k^2\sin\operatorname{am}a\sin\operatorname{am}b[\sin\operatorname{am}(a+b)-\sin\operatorname{am}(a-b)]u+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}b\sin^2\operatorname{am}(u-a)}{1-k^2\sin^2\operatorname{am}b\sin^2\operatorname{am}(u+a)}$$
$$(8.)\qquad \Pi(u,a+b)-\Pi(u,a-b)-2\Pi(u,b)$$
$$-k^2\sin\operatorname{am}a\sin\operatorname{am}b[\sin\operatorname{am}(a+b)-\sin\operatorname{am}(a-b)]u+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u-b)}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+b)}.$$

Having put $b=0$, it arises from (4.), (5.):

$$(9.)\qquad \Pi(u+v,a)+\Pi(u-v,a)-2\Pi(u,a)=\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}v\sin^2\operatorname{am}(u-a)}{1-k^2\sin^2\operatorname{am}v\sin^2\operatorname{am}(u+a)}$$
$$(10.)\qquad \Pi(u+v,a)-\Pi(u-v,a)-2\Pi(v,a)=\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}(v-a)}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}(v+a)}.$$

2.7 Reductions of the Expressions $Z(iu)$, $\Theta(iu)$ to a real Argument. The general reduction of elliptic Integrals of the third Kind, in which the Arguments both of the Amplitude and the Parameter are imaginary

56.

We return to the analysis of the functions Z, Θ whose extraordinary use in our theory we probed in the preceding paragraphs. Let us ask about the reduction of the expressions $Z(iu)$, $\Theta(iu)$ to a real argument. We pursue the same at first using Legendre's notation, then we will use our notation.

We know that in the elements, § 19 p. 85, the following equations hold at the same time:

$$\sin\varphi == i\tan\Psi, \qquad \frac{d\varphi}{\Delta(\varphi)} = \frac{id\psi}{\Theta(\psi,k')}, \qquad F(\varphi) = iF(\psi,k').$$

Hence, it is:

$$d\Delta(\varphi) = \frac{id\psi(1+kk\tan^2\psi)}{\Delta(\psi,k')} = \frac{id\psi\Delta(\psi,k')}{\cos^2\psi},$$

whence after an integration:

$$\int\limits_0^\varphi \Delta(\varphi)d\varphi = i\left\{\tan\psi\Delta(\psi,k') + \int\limits_0^\psi \frac{k'k'\sin^2\psi}{\Delta(\psi,k')}d\psi\right\}$$

or:

$$(1.) \qquad E(\varphi) = i[\tan\psi\Delta(\psi,k') + F(\psi,k') - E(\psi,k')].$$

By multiplying by $\frac{d\varphi}{\Delta(\varphi)} = \frac{id\psi}{\Delta(\psi,k')}$ and by integrating we find:

$$(2.) \qquad \int\limits_0^\varphi \frac{E(\varphi)}{\Delta(\varphi)}d\varphi = \ln\cos\psi - \frac{1}{2}\left\{F(\psi,k')\right\}^2 + \int\limits_0^\psi \frac{E(\psi,k')}{\Delta(\psi,k')}d\psi.$$

From equation (1.) it follows:

$$\frac{F^{\mathrm{I}}E(\varphi)E^{\mathrm{I}}F(\varphi)}{i} = F^{\mathrm{I}}\tan(\psi)\Delta(\psi,k') - [F^{\mathrm{I}}E(\psi,k') - (E^{\mathrm{I}} - F^{\mathrm{I}})F(\psi,k')].$$

Now, note the extraordinary theorem due to Legendre (pag. 61):

$$F^{\mathrm{I}}E^{\mathrm{I}}(k') + F^{\mathrm{I}}(k')E^{\mathrm{I}} - F^{\mathrm{I}}F^{\mathrm{I}}(k') = \frac{\pi}{2},$$

whence:

$$F^{\mathrm{I}}E(\psi,k') + (E^{\mathrm{I}} - F^{\mathrm{I}})F(\psi,k') = \frac{F^{\mathrm{I}}}{F^{\mathrm{I}}(k')}[F^{\mathrm{I}}(k')E(\psi,k') - E^{\mathrm{I}}(k')F(\psi,k')] + \frac{\pi F(\psi,k')}{2F^{\mathrm{I}}(k')},$$

and hence:

$$(3.)\qquad \frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{iF^{\mathrm{I}}}=\tan\psi\Delta(\psi,k')-\frac{F^{\mathrm{I}}(k')E(\psi,k')-E^{\mathrm{I}}(k')F(\psi,k')}{F^{\mathrm{I}}(k')}-\frac{\pi F(\psi,k')}{2F^{\mathrm{I}}F^{\mathrm{I}}(k')}.$$

From our notation it was:

$$\varphi=\mathrm{am}(iu),\quad \psi=\mathrm{am}(u,k'),\quad E(\varphi)=iu,\quad F(\psi,k')=u;$$

further,

$$\frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}}=Z(iu,k),\quad \frac{F^{\mathrm{I}}(k')E(\psi,k')-E^{\mathrm{I}}(k')F(\psi,k')}{F^{\mathrm{I}}(k')}=Z(u,k'),$$

whence equation (3.) is also represented this way:

$$(4.)\qquad iZ(iu,k)=-\tan\mathrm{am}(u,k')\Delta(u,k')+\frac{\pi u}{2KK'}+Z(u,k').$$

Hence, it arises from integration:

$$\int\limits_0^u idu Z(iu,k)=\ln\cos\mathrm{am}(u,k')+\frac{\pi uu}{4KK'}+\int\limits_0^u Z(u,k')du,$$

or, because it is $\int_0^u duZ(u)=\ln\frac{\Theta(u)}{\Theta(0)}$:

$$(5.)\qquad \frac{\Theta(iu,k)}{\Theta(0,k)}=e^{\frac{\pi uu}{4KK'}}\cos\mathrm{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')}.$$

Formulas (4.), (5.) reduce the functions $Z(iu)$, $\Theta(iu)$ to a real argument.

57.

In (5.) of the preceding § change u into $u+2K'$, it arises:

$$\frac{\Theta(iu+2iK')}{\Theta(0)}=-e^{\frac{\pi(u+2K')^2}{4KK'}}\cos\mathrm{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k)}=-e^{\frac{\pi(K'+u)}{K}}\frac{\Theta(iu)}{\Theta(0)},$$

or having put u instead of iu:

$$(1.)\qquad \Theta(u+2iK)=-e^{\frac{\pi(K'-iu)}{K}}\Theta(u).$$

In (5.) of the preceding § put $u+K'$ instead of u: Because it is:

$$\cos\operatorname{am}(u+K',k') = -\frac{k\sin\operatorname{am}(u,k'}{\Delta\operatorname{am}(u,k'}$$
$$\Theta(u+K',k') = \frac{\Delta\operatorname{am}(u,k')}{\sqrt{k}}\Theta(u,k'),$$

confer § 53 (9.), it arises:

$$\begin{aligned}\frac{\Theta(iu+iK')}{\Theta(0)} &= -e^{\frac{\pi(u+K')^2}{4KK'}}\sqrt{k}\sin\operatorname{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')}\\ &= -e^{\frac{\pi(2u+K')^2}{4K}}\sqrt{k}\tan\operatorname{am}(u,k')\frac{\Theta(iu)}{\Theta(0)},\end{aligned}$$

whence having put u instead of iu again:

$$(2.)\qquad \Theta(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}\sqrt{k}\sin\operatorname{am}u\Theta(u).$$

Having taken logarithm and by differentiating it arises from (1.) and (2.):

$$(3.)\qquad Z(u+2iK') = \frac{-i\pi}{K} + Z(u)$$
$$(4.)\qquad Z(u+iK') = \frac{-i\pi}{2K} + \cot\operatorname{am}u\Delta\operatorname{am}u + Z(u).$$

Having put $u=0$, it is from (1.) − (4.):

$$(5.)\qquad \begin{cases}\Theta(2iK') = -e^{\frac{\pi K'}{K}}\Theta(0), & \Theta(iK') = 0\\ Z(2iK') = \dfrac{-i\pi}{K}, & Z(iK') = \infty\end{cases}$$

Formulas (1.), (2.) find an extraordinary confirmation from the nature of the infinite products, into which we expanded the function Θ:

$$(6.)\qquad \begin{aligned}\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} &= \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}\\ &= \frac{[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots]}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}\end{aligned}$$

For, if x is changed into $x + \frac{i\pi K'}{K}$ having done which e^{ix} goes over into qe^{ix} the product:

$$[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots]$$

goes over into this one:

$$\frac{-1}{qe^{2ix}}[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots],$$

whence:

$$(7.) \qquad \Theta\left(\frac{2Kx}{\pi} + 2iK'\right) = -\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{q^{2ix}}.$$

On the other hand, having changed x into $x + \frac{i\pi K'}{2K}$, e^{ix} goes over into $\sqrt{q}e^{ix}$, whence the product:

$$[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots]$$

goes over into this one:

$$\begin{aligned} &(1-e^{-2ix})[(1-q^2e^{2ix})(1-q^4e^{2ix})\cdots][(1-q^2e^{-2ix})(1-q^4e^{-2ix})\cdots] \\ = &\frac{i}{e^{ix}}\cdot 2\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots \end{aligned}$$

But in § 36 we gave the formula:

$$\sin\operatorname{am}\frac{2Kx}{\pi} = \frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

whence we see that it will be:

$$(8.) \qquad \Theta\left(\frac{2Kx}{\pi} + iK'\right) = \frac{i\sqrt{k}\sin\operatorname{am}\frac{2Kx}{\pi}\Theta\left(\frac{2Kx}{\pi}\right)}{\sqrt[4]{q}e^{ix}}.$$

But, formulas (7.), (8.) having put $\frac{2Kx}{\pi} = u$ agree with formulas (1.), (2.)

From formula (9.) § 53:

$$\Theta(u+K) = \frac{\Delta \operatorname{am} u}{\sqrt{k'}} \cdot \Theta(u),$$

having put iu instead of u, it follows:

$$\Theta(iu+K) = \frac{\Delta \operatorname{am}(u,k')}{\sqrt{k'} \cos \operatorname{am}(u,k')} \cdot \Theta(iu),$$

whence from (5.) § 56:

$$\frac{\Theta(iu+K)}{\Theta(0)} = \frac{1}{\sqrt{k'}} e^{\frac{\pi uu}{4KK'}} \Delta \operatorname{am}(u,k') \cdot \frac{\Theta(u,k')}{\Theta(0,k')}$$

or from the aforementioned formula (9.) § 53:

$$(9.) \qquad \frac{\Theta(iu+K)}{\Theta(0)} = \sqrt{\frac{k}{k'}} e^{\frac{\pi uu}{4KK'}} \frac{\Theta(u+K',k')}{\Theta(0,k')}.$$

Hence, by taking logarithm and differentiating we obtain:

$$(10.) \qquad iZ(iu+K) = \frac{\pi u}{2KK'} + Z(u+K',k').$$

58.

The formulas found in §§ 56 and 57 have a simple application to the analysis of the functions Π in the cases in which the arguments either of the amplitude or of the parameter or even of both are imaginary.

At first, let us demonstrate that the expression $\Pi(u, a+iK')$ can be reduced to $\Pi(u,a)$ whence it is clear that having put $n = -k^2 \sin^2 \operatorname{am} a$ the integrals:

$$\int_0^\varphi \frac{d\varphi}{(1+n\sin^2\varphi)\Delta(\varphi)}, \quad \int_0^\varphi \frac{d\varphi}{\left(1+\frac{k^2}{n}\sin^2\varphi\right)\Delta(\varphi)}$$

depend on each other; this is an extraordinary theorem given by Legendre in cap. XV.

We found:

$$\Pi(u, a+iK') = uZ(a+iK') + \frac{1}{2}\ln\frac{\Theta(a-u+iK')}{\Theta(a+u+iK')}.$$

But it is from (2.), (4.) § 57:

$$\frac{\Theta(a-u+iK')}{\Theta(a+u+iK')} = e^{\frac{i\pi u}{K}} \frac{\sin\operatorname{am}(a-u)}{\sin\operatorname{am}(a-u)} \cdot \frac{\Theta(a-u)}{\Theta(a+u)}$$

$$uZ(a+iK') = -\frac{i\pi u}{2K} + u\cot\operatorname{am} a\Delta\operatorname{am} a + uZ(a),$$

whence while the terms $\frac{i\pi u}{2K}, -\frac{i\pi u}{2K}$ cancel:

$$\text{(1.)} \qquad \Pi(u, a+iK') = \Pi(u,a) + u\cot\operatorname{am} a\Delta\operatorname{am} a + \frac{1}{2}\ln\frac{\sin\operatorname{am}(a-u)}{\sin\operatorname{am}(a+u)}.$$

Let in this formula put ia instead of a, it is:

$$\cot\operatorname{am} ia\Delta\operatorname{am} ia = \frac{-i\Delta\operatorname{am}(a.k')}{\sin\operatorname{am}(a,k')\cos\operatorname{am}(a,k')}$$

$$\frac{\sin\operatorname{am}(ia-u)}{\sin\operatorname{am}(ia+u)} = \frac{\Delta\operatorname{am} u - \cot\operatorname{am} ia\Delta\operatorname{am} ia\tan\operatorname{am} u}{\Delta\operatorname{am} u + \cot\operatorname{am} ia\Delta\operatorname{am} ia\tan\operatorname{am} u},$$

or having put for the sake of gravity:

$$\frac{\Delta\operatorname{am}(a,k')}{\sin\operatorname{am}(a,k')\cos\operatorname{am}(a,k')} = \sqrt{\alpha},$$

it is:

$$\frac{\sin\operatorname{am}(ia-u)}{\sin\operatorname{am}(ia+u)} = \frac{\Delta\operatorname{am} u + i\sqrt{\alpha}\tan\operatorname{am} u}{\Delta\operatorname{am} u - i\sqrt{\alpha}\tan\operatorname{am} u},$$

whence (1.) goes over into:

$$\text{(2.)} \qquad \frac{\Pi(u, ia+iK') - \Pi(u, ia)}{i} = -\sqrt{\alpha}\cdot u + \arctan\frac{\sqrt{\alpha}\tan\operatorname{am} u}{\Delta\operatorname{am} u},$$

which agrees with formula (f') exhibited by Legendre.

59.

We obtain other formulas, fundamental for the reduction of an imaginary argument to a real one, from (9.), (10.). First, I mention that one of those by means of which imaginary arguments of both the amplitude and the parameter are reduced to real arguments.

$$(1.) \qquad \Pi(iu, ia+K) = \Pi(u, a+K', k'),$$

which is demonstrated this way. For, it is:

$$\Pi(iu, ia+K) = iuZ(ia+K) + \frac{1}{2}\ln\frac{\Theta(ia-iu+K)}{\Theta(ia+iu+K)};$$

further, from (10.) § 57:

$$iuZ(ia+K) = \frac{\pi ua}{2KK'} + uZ(a+K', k'),$$

from (9.) § 57:

$$\begin{aligned}
\frac{\Theta(ia-iu+K)}{\Theta(0,k)} &= \sqrt{\frac{k}{k'}}e^{\frac{\pi(a-u)^2}{4KK'}}\frac{\Theta(a-u+K',k')}{\Theta(0,k')} \\
\frac{\Theta(ia+iu+K)}{\Theta(0,k)} &= \sqrt{\frac{k}{k'}}e^{\frac{\pi(a+u)^2}{4KK'}}\frac{\Theta(a+u+K',k')}{\Theta(0,k')},
\end{aligned}$$

whence:

$$\frac{\Theta(ia-iu+K)}{\Theta(ia+iu+K)} = e^{\frac{-\pi au}{KK'}}\frac{\Theta(a-u+K',k')}{\Theta(a+u+K',k')}$$

and hence cancelling the terms $\frac{\pi ua}{2KK'}$, $-\frac{\pi ua}{2KK'}$:

$$\Pi(iu, ia+K) = uZ(a+K',k') + \frac{1}{2}\ln\frac{\Theta(a-u+K',k')}{\Theta(a+u+K',k')} = \Pi(u, a+K', k'),$$

which was to be demonstrated.

Having in (1.) changed a to $-ia$ it arises:

$$(2.) \qquad \Pi(iu, a+K) = -\Pi(u, ia+K', k').$$

Formula (1.) is also easily proved considering the integral itself by means of which we defined the function Π:

$$\Pi(u,a) = \int\limits_0^u \frac{k^2 \sin\operatorname{am} a \cos\operatorname{am} a \Delta\operatorname{am} a \sin^2\operatorname{am} u}{1-k^2\sin^2\operatorname{am} a\sin^2\operatorname{am} u}du,$$

whence:

$$\Pi(iu, ia+K) = \int_0^u \frac{ik^2 \sin\operatorname{am}(ia+K)\cos\operatorname{am}(ia+K)\Delta\operatorname{am}(ia+K)\sin^2\operatorname{am} iu}{1-k^2\sin^2\operatorname{am}(ia+K)\sin^2\operatorname{am} u} du.$$

For, it is from the formulas of § 19:

$$\begin{aligned}
\sin\operatorname{am}(ia+K) &= +\sin\operatorname{coam} ia = \frac{\Delta\operatorname{coam}(a,k')}{k} &&= \frac{\Delta\operatorname{am}(a+K',k')}{k}\\
\cos\operatorname{am}(ia+K) &= -\cos\operatorname{coam} ia = \frac{-ik'}{k}\cos\operatorname{coam}(a,k') &&= \frac{ik'}{k}\cos\operatorname{am}(a+K',k')\\
\Delta\operatorname{am}(ia+K) &= \Delta\operatorname{coam} ia = k'\sin\operatorname{coam}(a,k') &&= k'\sin\operatorname{am}(a+K',k'),
\end{aligned}$$

whence:

$$\begin{aligned}
&+ikk\sin\operatorname{am}(ia+K)\cdot\cos\operatorname{am}(ia+K)\cdot\Delta\operatorname{am}(ia+K)\\
&= -k'k'\sin\operatorname{am}(a+K',k')\cos\operatorname{am}(a+K',k')\Delta\operatorname{am}(a+K',k').
\end{aligned}$$

Further, it is:

$$\frac{\sin^2\operatorname{am} iu}{1-k^2\sin^2\operatorname{am}(ia+K)\sin^2\operatorname{am} iu} = \frac{-tan^2\operatorname{am}(u,k')}{1+\Delta\operatorname{am}(a+K',k')\tan^2\operatorname{am}(u,k')}$$

$$= \frac{-\sin^2\operatorname{am}(u,k')}{\cos^2\operatorname{am}(u,k')+\Delta^2\operatorname{am}(a+K',k')\sin^2\operatorname{am}(u,k')} = \frac{-\sin^2\operatorname{am}(u,k')}{1-k'k'\sin^2\operatorname{am}(a+K',k')\sin^2\operatorname{am}(u,k')},$$

whence:

$$\Pi(iu, ia+K) = \int_0^u \frac{k'k'\sin\operatorname{am}(a+K',k')\cos\operatorname{am}(a+K',k')\Delta\operatorname{am}(a+K',k')\sin^2\operatorname{am}(u,k')}{1-k'k'\sin^2\operatorname{am}(a+K',k')\sin^2\operatorname{am}(u,k')} du,$$

or:

$$\Pi(iu, ia+K) = \Pi(u, a+K', k'),$$

what was to be proved.

From formulas (9.), (10.) of § 57 in the same way as (1.) we can prove the following formula which teaches that two functions of an imaginary argument of the parameter of which the one the one is the complement of the other modulus can be reduced to each other:

$$(3.)\qquad i\Pi(u, ia+K)+i\Pi(a, iu+K', k') = \frac{\pi a u}{2KK'}+uZ(a+K', k')+aZ(u+K, k).$$

For, it is:

$$\begin{aligned} i\Pi(u, ia+K) &= iuZ(ia+K) &&+\frac{i}{2}\ln\frac{\Theta(ia+K-u)}{\Theta(ia+K+u)} \\ i\Pi(a, iu+K', k') &= iaZ(iu+K', k') &&+\frac{i}{2}\ln\frac{\Theta(iu+K'-a, k')}{\Theta(iu+K'+a, k')}. \end{aligned}$$

Now, it is:

$$\begin{aligned} \frac{\Theta(ia+K-u)}{\Theta(0)} &= \frac{\Theta(i(a+iu)+K)}{\Theta(0)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi(a+iu)^2}{4KK'}}\frac{\Theta(a+iu+K', k')}{\Theta(0, k')} \\ \frac{\Theta(ia+K+u)}{\Theta(0)} &= \frac{\Theta(i(a-iu)+K)}{\Theta(0)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi(a-iu)^2}{4KK'}}\frac{\Theta(a-iu+K', k')}{\Theta(0, k')} \end{aligned}$$

whence, because it is $\Theta(u+K) = \Theta(K-u)$:

$$\frac{\Theta(ia+K-u)}{\Theta(ia+K+u)} = e^{\frac{i\pi a u}{KK'}}\frac{\Theta(iu+K'+a, k')}{\Theta(iu+K'-a, k')},$$

and hence:

$$\frac{i}{2}\ln\frac{\Theta(ia+K-u)}{\Theta(ia+K+u)}+\frac{i}{2}\ln\frac{\Theta(iu+K'-a, k')}{\Theta(iu+K'+a, k')} = -\frac{\pi a u}{2KK'}.$$

Further, it is:

$$\begin{aligned} iuZ(ia+K) &= \frac{\pi a u}{2KK'}+uZ(a+K', k') \\ iaZ(iu+K', k') &= \frac{\pi a u}{2KK'}+aZ(u+K, k), \end{aligned}$$

whence:

$$i\Pi(u, ia + K) + i\Pi(a, iu + K', k') = \frac{\pi au}{2KK'} + uZ(a + K', k') + aZ(u + K, k),$$

q.d.e.

60.

It is clear from the formulas:

$$\begin{aligned} \sin\operatorname{am}(K + iu) &= \frac{1}{k}\Delta\operatorname{coam}(u, k') \\ \sin\operatorname{am}(u + iK') &= \frac{1}{k} \cdot \frac{1}{\sin\operatorname{am} u}, \end{aligned}$$

that the argument u which as $\sin\operatorname{am} u$ grows from 0 to 1 increases from 0 to K, if $\sin\operatorname{am} u$ goes from 1 to $\frac{1}{k}$, takes an imaginary value of the form $K + iv$ such that at the same time v increases from 0 to K'; after this, while $\sin\operatorname{am} u$ grows from $\frac{1}{k}$ to ∞, u takes the form $v + iK'$ such that at the same time v decreases from K to 0.

Hence, we see that, if in the elliptic integrals of the third kind, which is contained in this scheme:

$$\int_0^\varphi \frac{d\varphi}{(1 + n\sin^2\varphi)\Delta(\varphi)},$$

one puts, as we did, $n = -\sin^2\sin^2\operatorname{am} a$, if n is negative

between	$+0$	and	$-kk$,	one has to put	$n = -k^2\sin^2\operatorname{am} a$
between	$-kk$	and	-1,	one has to put	$n = -k^2\sin^2\operatorname{am}(ia + K)$
between	-1	and	$-\infty$,	one has to put	$n = -k^2\sin^2\operatorname{am}(a + iK'$,

while a denotes a real quantity. Further, because it is $-kk\sin^2\operatorname{am} ia = kk\tan\tan^2\operatorname{am}(a, k')$, it becomes clear, if n is an arbitrary positive number, one has to put:

$$n = -kk\sin^2\operatorname{am} ia,$$

Hence, we obtained four classes of elliptic integrals of the third kind which correspond to the schemes which tale the following arguments:

$$(1) \quad a, \qquad 2) \quad ia+K, \qquad 3) \quad a+iK', \qquad 4) \quad ia,$$

of which the first three correspond to a negative n, the fourth to a positive n.

But, by means of formula (1.) § 58 we see that the function $\Pi(u, a+iK')$ can be reduced to $\Pi(u,a)$, or the third class, in which n is between -1 and $-\infty$, can be reduced to the first, in which n is between 0 and $-kk$. Further, form formula (11.) of § 53 we see that the function $\Pi(u, ia)$ can always be reduced to $\Pi(u, ia+K)$ or the fourth class in which n is positive to the second in which n is negative between $-kk$ and -1. Hence, we now obtained the theorem that *the propounded integral:*

$$\int\limits_0^\varphi \frac{d\varphi}{(1+n\sin^2\varphi)\Delta(\varphi)},$$

whatever real positive or negative number n is, can always be reduced to a similar integral, in which n is negative and between 0 *and* 1. This is the extraordinary discovery of Legendre.

But now let us consider the general case, in which both the amplitude and the parameter have an arbitrary imaginary form: It is clear that this case contains the expression:

$$\Pi(u+iv, a+ib),$$

u,v, a, b denoting real numbers. But from the formulas of § 55 we see that an expression of such a kind can be reduced to these four:

$$1) \quad \Pi(u,a), \qquad 2) \quad \Pi(iv,ib), \qquad 3) \quad \Pi(u,ib), \qquad 4) \quad \Pi(iv,a),$$

or, if it pleases, to these four:

$$1) \quad \Pi(u,a-K), \qquad 2) \quad \Pi(iv,ib+K), \qquad 3) \quad \Pi(u,ib+K), \qquad 4) \quad \Pi(iv,a-K).$$

For, in general the expression $\Pi(u+v, a+b)$ reduces to $\Pi(u,a)$, $\Pi(v,b)$, $\Pi(u,b)$, $\Pi(v,a)$ from which the four propounded formulas arise, if instead of

v one puts iv, but instead of a,b $a-K$, $K+ib$. Further, it is from the formulas (1.), (2.) of § 59:

$$\begin{aligned}
\Pi(iv, ib+K) &= +\Pi(v, b+K', k') \\
\Pi(iv, a-K) &= -\Pi(v, ia+K', k'),
\end{aligned}$$

whence expressions 1), 2) reduce to the first class $\Pi(u,a)$, the expressions 3), 4) reduce to the second class $\Pi(u, ia+K)$; this gives us the following

Theorem

The propounded integral of the form

$$\int_0^{\varphi} \frac{d\varphi}{(1+n\sin^2\varphi)\Delta(\varphi)},$$

whatever n and φ are, either real or imaginary, can be reduced to similar integrals in which φ is real and n is real and between 0 *and* -1.

This theorem is also due to Legendre except that he considered only real amplitudes.

By means of formulas (4.), (5.) of § 55 $\Pi(u+v, a+b) + \Pi(u-v, a-b)$ is reduced to $\Pi(u,a)$ and $\Pi(v,b)$, $\Pi(u+v, a+b) - \Pi(u-v, a-b)$ is reduced to $\Pi(u,b)$ and $\Pi(v,a)$. Hence, it is clear that having put:

$$\begin{aligned}
\Pi(u+iv, a+ib) + \Pi(u-iv, a-ib) &= L, \\
\frac{\Pi(u+iv, a+ib) - \Pi(u-iv, a-ib)}{i} &= M,
\end{aligned}$$

L depends on the functions $\Pi(u, a-K)$, $\Pi(iv, ib+K)$, M depends on the functions $\Pi(u, ib+K)$, $\Pi(iv, a-K)$. and hence L reduces to the first class, M reduces to the second class.

These are the foundations of the theory of the elliptic integrals of the third kind, deduced from new principles. We will see others below.

2.8 Elliptic Functions are rational Functions. On the Functions H, Θ which take the place of the numerator and the denominator.

61.

The expansions exhibited in § 35 reveal the genuine nature of elliptic functions, of course that they are rational functions, and as we already know from the elements, that they vanish and become infinite for innumerable different values of the argument. We have already in the preceding paragraphs been led to the function which function which constitutes the denominator of the fraction, into which we expanded the latter

$$\sin\operatorname{am}\frac{2kx}{\pi}=\frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

I mean the function:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)}=\frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots]^2}.$$

Now, let us also denote the numerator by a particular character, and let us put:

$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)}=\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{[(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots]^2}.$$

it will be:

$$\sin\operatorname{am}\frac{2Kx}{\pi}=\frac{1}{\sqrt{k}}\cdot\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}.$$

Having recalled the expansions given in § 36, we find:

$$\begin{aligned}\cos\operatorname{am}\frac{2Kx}{\pi}&=\sqrt{\frac{k'}{k}}\cdot\frac{H\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}\\ \Delta\operatorname{am}\frac{2Kx}{\pi}&=k'\cdot\frac{\Theta\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)},\end{aligned}$$

whence having put $\frac{2Kx}{\pi} = u$:

$$(1.) \quad \sin\operatorname{am} u = \frac{1}{\sqrt{k}} \cdot \frac{H(u)}{\Theta(u)}; \quad \cos\operatorname{am} u = \sqrt{\frac{k'}{k}} \cdot \frac{H(u+K)}{\Theta(u)}; \quad \Delta\operatorname{am} u = \sqrt{k'} \cdot \frac{\Theta(u+K)}{\Theta(u)}.$$

Hence the special formulas follow:

$$(2.) \qquad \Theta(K) = \frac{\Theta(0)}{\sqrt{k'}}, \quad H(K) = \sqrt{\frac{k}{k'}}\Theta(0).$$

Having put $H'(u) = \frac{dH(u)}{du}$, because it is:

$$H'(u) = \sqrt{k}\cos\operatorname{am} u \ \Delta\operatorname{am} u\Theta(u) + \sqrt{k}\sin\operatorname{am} u \ \Theta'(u),$$

we obtain for the values $u = 0$, $u = K$:

$$(3.) \quad H'(0) = \sqrt{k}\Theta(0) = \frac{H(K)\Theta(0)}{\Theta(K)}; \quad H'(K) = \sqrt{k}\Theta'(K) = 0.$$

From (2.) it also follows:

$$(4.) \qquad \sqrt{k} = \frac{H(K)}{\Theta(K)}; \qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta(K)}.$$

Moreover, it is:

$$(5.) \qquad \Theta(u+2K) = \Theta(-u) = \Theta(u)$$
$$(6.) \qquad H(u+2K) = H(-u) = -H(u); \quad H(u+4K) = H(u);$$

From formula (2.) of § 57:

$$\Theta(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}\sqrt{k}\sin\operatorname{am} u \ \Theta(u)$$

it follows:

$$(7.) \qquad \Theta(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}H(u).$$

Having in this formula changed u into $u + iK'$ and recalled (1.) of § 57:

$$(8.) \qquad \Theta(u+2iK') = -e^{\frac{\pi(K'-iu)}{K}}\Theta(u),$$

it arises:

$$(9.)\qquad H(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}\Theta(u),$$

whence, having again changed u to $u+iK'$, from (7.):

$$(10.)\qquad H(u+2iK') = -e^{\frac{\pi(K'-iu)}{K}}H(u).$$

From the formulas (7.) $-$ (10.) one can derive the more general ones:

$$\begin{aligned}
&(11.)\qquad e^{\frac{\pi uu}{4KK'}}\Theta(u) = (-1)^m e^{\frac{\pi(u+2miK')}{4KK'}}\Theta(u+2miK')\\
&(12.)\qquad e^{\frac{\pi uu}{4KK'}}H(u) = (-1)^m e^{\frac{\pi(u+2miK')}{4KK'}}H(u+2miK')\\
&(13.)\qquad e^{\frac{\pi uu}{4KK'}}H(u) = (-i)^{2m+1} e^{\frac{\pi(u+(2m+1)iK')}{4KK'}}\Theta(u+(2m+1)iK')\\
&(14.)\qquad e^{\frac{\pi uu}{4KK'}}\Theta(u) = (-i)^{2m+1} e^{\frac{\pi(u+(2m+1)iK')}{4KK'}}H(u+(2m+1)iK')
\end{aligned}$$

From (12.), (13.) it is:

$$(15.)\qquad \Theta((2m+1)iK') = 0, \qquad H(2miK') = 0.$$

Formulas (5.), (6.) show that the functions $\Theta(u)$, $H(u)$ having changed u to $u+4K$, formulas (11.), (12.) that the functions

$$e^{\frac{\pi uu}{4KK'}}\Theta(u), \qquad e^{\frac{\pi uu}{4KK'}}H(u),$$

having changed u into $u+4iK'$, remain unchanged; hence, those have the real period in common with the elliptic functions, those have the other imaginary period in common with the elliptic functions.

From formula (5.) § 56:

$$\frac{\Theta(iu,k)}{\Theta(0,k)} = e^{\frac{\pi uu}{4KK'}}\cos\operatorname{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')}$$

it follows:

$$\frac{H(iu,k)}{\Theta(0,k)} = \sqrt{k}\sin\operatorname{am}(iu,k)\frac{\Theta(iu,k)}{\Theta(0,k)} = ie^{\frac{\pi uu}{4KK'}}\sqrt{k}\sin\operatorname{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')},$$

whence from (1.):

$$(16.)\qquad \frac{\Theta(iu,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{H(u+K',k')}{\Theta(0,k')}$$

$$(17.)\qquad \frac{H(iu,k)}{\Theta(0,k)} = i\sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{H(u,k')}{\Theta(0,k')}.$$

From (16.) it follows having changed u to iu and commuted k and k':

$$(18.)\qquad \frac{H(iu+K,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{\Theta(u,k')}{\Theta(0,k')},$$

to which we add (9.) § 57:

$$(19.)\qquad \frac{\Theta(iu+K,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{\Theta(u+K',k')}{\Theta(0,k')}.$$

From the formula found above:

$$\Theta(u+v)\Theta(u-v) = \frac{\Theta^2(u)\Theta^2(v)}{\Theta^2(0)}(1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v)$$

it follows:

$$(20.)\qquad \Theta(u+v)\Theta(u-v) = \frac{\Theta^2(u)\Theta^2(v)-H^2(u)H^2(v)}{\Theta^2(0)}.$$

Having multiplied the formula by:

$$k\sin\operatorname{am}(u+v)\sin\operatorname{am}(u-v) = \frac{k\sin^2\operatorname{am}u - k\sin^2\operatorname{am}v}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v} = \frac{H^2(u)\Theta^2(v)-\Theta^2(u)H^2(v)}{\Theta^2(u)\Theta^2(v)-H^2(u)H^2(v)},$$

it arises:

$$(21.)\qquad H(u+v)H(u-v) = \frac{H^2(u)\Theta^2(v)-\Theta^2(u)H^2(v)}{\Theta^2(0)}.$$

2.9 On the expansion of the Functions H, Θ into series. Third Expansion of the elliptic Functions.

62.

Let us expand the functions:

$$\frac{\Theta\left(\frac{2kx}{\pi}\right)}{\Theta(0)}=\frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}$$
$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)}=\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}$$

into series:

$$\frac{\Theta\left(\frac{2kx}{\pi}\right)}{\Theta(0)}=A-2A'\cos 2x+2A''\cos 4x-2A'''\cos 6x+2A^{\text{IV}}\cos 8x-\cdots$$
$$\frac{H\left(\frac{2kx}{\pi}\right)}{\Theta(0)}=2\sqrt[4]{q}[B'\sin x-2B''\sin 3x+2B'''\sin 5x-2B^{\text{IV}}\sin 7x+\cdots].$$

We obtain the determination of A, A', A'', $A'''\cdots$; B', B'', B''', $B^{\text{IV}},\cdots$ by means of equations $(7.)-(10.)$ of the preceding § which having put $U=\frac{2Kx}{\pi}$, $q=e^{\frac{-\pi K'}{K}}$ go over into the following:

$$\begin{aligned}
\Theta\left(\frac{2Kx}{\pi}\right)&=-qe^{2ix}\Theta\left(\frac{2Kx}{\pi}+2iK'\right)\\
H\left(\frac{2Kx}{\pi}\right)&=-qe^{2ix}H\left(\frac{2Kx}{\pi}+2iK'\right)\\
i\Theta\left(\frac{2Kx}{\pi}\right)&=+\sqrt[4]{q}e^{ix}H\left(\frac{2Kx}{\pi}+iK'\right)\\
iH\left(\frac{2Kx}{\pi}\right)&=+\sqrt[4]{q}e^{ix}\Theta\left(\frac{2Kx}{\pi}+iK'\right).
\end{aligned}$$

For this aim, we exhibit the propounded expansions this way:

$$\begin{aligned}
\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)}=A&-A'e^{2ix}+A''e^{4ix}-A'''e^{6ix}+A^{\text{IV}}e^{8ix}-\cdots\\
&-A'e^{-2ix}+A''e^{-4ix}-A'''e^{-6ix}+A^{\text{IV}}e^{-8ix}-\cdots
\end{aligned}$$

$$\frac{iH\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \sqrt[4]{q}[B'e^{ix} - B''e^{3ix} + B'''e^{5ix} - B^{\mathrm{IV}}e^{7ix} + \cdots]$$
$$- \sqrt[4]{q}[B'e^{-ix} - B''e^{-3ix} + B'''e^{-5ix} - B^{\mathrm{IV}}e^{-7ix} + \cdots]$$

Having changed x to $x - i\ln q$ e^{mix} goes over into $q^m e^{mix}$, e^{-mix} goes over into $\frac{e^{-mix}}{q^m}$; further, $\Theta\left(\frac{2Kx}{\pi}\right)$, $H\left(\frac{2Kx}{\pi}\right)$ go over into $\Theta\left(\frac{2Kx}{\pi} + 2iK'\right)$, $H\left(\frac{2Kx}{\pi} + 2iK'\right)$. Hence, we obtain:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = -qe^{2ix} \cdot \frac{\Theta\left(\frac{2Kx}{\pi} + 2iK'\right)}{\Theta(0)}$$

$$= \frac{A'}{q} - Aqe^{2ix} + A'q^3e^{4ix} - A''q^5e^{6ix} + A'''q^7e^{8ix} - \cdots$$
$$- \frac{A''}{q^3}e^{-2ix} + \frac{A''}{q^5}e^{-4ix} - \frac{A^{\mathrm{IV}}}{q^7}e^{-6ix} + \frac{A^{\mathrm{V}}}{q^9}e^{-8ix} - \cdots$$

$$\frac{iH\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = -qe^{2ix} \cdot \frac{iH\left(\frac{2Kx}{\pi} + 2iK'\right)}{\Theta(0)}$$

$$= \sqrt[4]{q}\left\{B'e^{ix} - B'q^2e^{3ix} + B''q^4e^{5ix} - B'''q^6e^{7ix} + \cdots\right\}$$
$$- \sqrt[4]{q}\left\{\frac{B''}{q^2}e^{-ix} - \frac{B'''}{q^4}e^{-3ix} + \frac{B^{\mathrm{IV}}}{q^6}e^{-5ix} - \frac{B^{\mathrm{V}}}{q^8}e^{-7ix} + \cdots\right\}.$$

Having compared those with the propounded expressions we find:

$$A' = Aq, \quad A'' = A'q^3, \quad A''' = A''q^5, \quad A^{\mathrm{IV}} = A'''q^7, \cdots,$$
$$B'' = B'q^2, \quad B''' = B''q^4, \quad B^{\mathrm{IV}} = B'''q^6, \quad B^{\mathrm{V}} = B^{\mathrm{IV}}q^8, \cdots,$$

and hence:

$$A' = Aq, \quad A'' = Aq^4, \quad A'''Aq^9, \quad A^{\mathrm{IV}} = Aq^{16}, \cdots,$$
$$B'' = B'q^2, \quad B''' = B'q^6, \quad B^{\mathrm{IV}}B'q^{12}, \quad B^{\mathrm{V}} = B'q^{20}, \cdots,$$

whence the looked-for expansions become:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = A[1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots]$$
$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = 2\sqrt[4]{q}B'[\sin x - q^2\sin 3x + q^{2\cdot 3}\sin 5x - q^{3\cdot 4}\sin 7x + q^{4\cdot 5}\sin 9x - \cdots]$$
$$= B'[2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots].$$

The found expansions could have been derived from each other by means of the formula:

$$iH\left(\frac{2Kx}{\pi}\right) = \sqrt[4]{q}e^{ix}\Theta\left(\frac{2Kx}{\pi} + iK'\right).$$

For, having found the series:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = A[1 - q(e^{2ix} + e^{-2ix}) + q^4(e^{4ix} + e^{-4ix}) - q^9(e^{6ix} + e^{-6ix}) + \cdots],$$

by changing x to $x - i\ln\sqrt{q}$ having done which e^{2mix}, e^{-2imx} go over into $q^m e^{2imx}$, $\frac{e^{-2mix}}{q^m}$, $\Theta\left(\frac{2Kx}{\pi}\right)$ into $\Theta\left(\frac{2Kx}{\pi} + iK'\right)$, and by multiplying by $\sqrt[4]{q}e^{ix}$ we obtain:

$$\frac{iH\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \sqrt[4]{q}e^{ix}\frac{\Theta\left(\frac{2Kx}{\pi} + iK'\right)}{\Theta(0)}$$
$$= A[\sqrt[4]{q}(e^{ix} - e^{-ix}) - \sqrt[4]{q^9}(e^{3ix} - e^{-3ix}) + \sqrt[4]{q^{25}}(e^{5ix} - e^{-5ix}) - \cdots]$$

or:

$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = A[2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots].$$

Additionally, we find by the this analysis:

$$B' = A.$$

63.

The determination of A demands particular artifices. Let us put, as is possible from the preceding:

$$
\begin{aligned}
&(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots \\
&= P(q)[1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots] \\
&\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots \\
&= P(q)[\sin x-q^{1\cdot 2}\sin 3x+q^{2\cdot 3}\sin 5x-q^{3\cdot 4}\sin 7x+q^{4\cdot 5}\sin 9x-\cdots];
\end{aligned}
$$

it is:

$$A=\frac{P(q)}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}.$$

The second expression remains unchanged if it is multiplied by the first, and after that it is put q^2 instead of q. Hence, we obtain the identity:

$$
\begin{aligned}
&P(q^2)P(q^2)[\sin x-q^4\sin 3x+q^{12}\sin 5x-q^{24}\sin 7x+\cdots] \\
&\times[1-2q^2\cos 2x+2q^8\cos 4x-2q^{18}\cos 6x+\cdots] \\
&= P(q)[\sin x.q^2\sin 3x+q^6\sin 5x-q^{12}\sin 7x+\cdots].
\end{aligned}
$$

Now, let us do the multiplication explicitly, such that everywhere instead of $2\sin mx\cos nx$ one writes $\sin(m+n)x+\sin(m-n)x$: It easily becomes clear that the coefficients of $\sin x$ in the expanded product will be:

$$1+q^2+q^6+q^{12}+q^{20}+\cdots,$$

such that arises:

$$\frac{P(q)}{P(q^2)P(q^2)}=1+q^2+q^6+q^{12}+q^{20}+\cdots$$

But we found from the second of the propounded formula having put $x=\frac{\pi}{2}$:

$$[(1+q^2)(1+q^4)(1+q^6)\cdots]^2=P(q)[1+q^2+q^6+q^{12}+q^{20}+\cdots],$$

whence:

$$\frac{P(q)P(q)}{P(q^2)P(q^2)} = [(1+q^2)(1+q^4)(1+q^6)\cdots]^2$$

or:

$$\begin{aligned}\frac{P(q)}{P(q^2)} &= (1+q^2)(1+q^4)(1+q^6)\cdots \\ &= \frac{(1-q^4)(1-q^8)(1-q^{12})\cdots}{(1-q^2)(1-q^4)(1-q^6)\cdots}.\end{aligned}$$

Hence, it finally arises:

$$\begin{aligned}A &= \frac{1}{(1-q^2)(1-q^4)(1-q^6)\cdots} \cdot \frac{1}{[(1-q)(1-q^3)(1-q^5)\cdots]^2} \\ &= \frac{(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}\end{aligned}$$

or from those expansions we found in § 36:

$$\frac{1}{A} = \sqrt{\frac{2k'K}{\pi}},$$

That quantity we left undetermined up to this point we now want to put $\Theta(0)$:

$$\Theta(0) = \frac{1}{A} = \sqrt{\frac{2k'K}{\pi}},$$

it is found:

$$(1.) \qquad \Theta\left(\frac{2Kx}{\pi}\right) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots,$$

$$(2.) \qquad H\left(\frac{2Kx}{\pi}\right) = 2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots$$

64.

It is convenient to investigate the identity which we proved in the last paragraph:

$$(1-2q\cos 2x+q^2)(1-2q^3\cos 3x+q^6)(1-2q^5\cos 2x+q^{10})\cdots$$
$$=\frac{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

on another way completely different from the preceding. For this aim, we want to mention the following two formulas as lemmas:

$$(1.)\qquad (1+qz)(1+q^3z)(1+q^5z)(1+q^7)\cdots$$
$$=1+\frac{qz}{1-q^2}+\frac{q^4z^2}{(1-q^2)(1-q^4)}+\frac{q^9z^3}{(1-q^2)(1-q^4)(1-q^6)}+\frac{q^{16}z^4}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)}+\cdots$$
$$(2.)\qquad \frac{1}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)\cdots}$$
$$=1+\frac{q}{1-q}\cdot\frac{z}{1-qz}+\frac{q^4}{(1-q)(1-q^2)}\cdot\frac{z^2}{(1-qz)(1-q^2z)}$$
$$+\frac{q^2}{(1-q)(1-q^2)(1-q^3)}\cdot\frac{z^3}{(1-qz)(1-q^2z)(1-q^3z)}+\cdots$$

For the demonstration of the first I observe that the expression:

$$(1+qz)(1+q^3z)(1+q^5z)(1+q^7z)\cdots,$$

having put q^2z instead of z and having multiplied by $(1+qz)$ remains unchanged; hence, having put:

$$(1+qz)(1+q^3z)(1+q^5z)\cdots=1+A'z+A''z^2+A'''z^3+\cdots,$$

it is found:

$$1+A'z+A''z^2+A'''z^3+\cdots=(1+qz)(1+A'q^2z+A''q^4z^2+A'''z^3+\cdots)$$

and hence, having done the expansion:

$$A'=q+q^2A',\quad A''=q^3A'+q^4A'',\quad A'''=q^5A''+q^6A''',\cdots$$

or:

$$A'=\frac{q}{1-q},\quad A''=\frac{q^3A'}{1-q^4},\quad A'''=\frac{q^5A''}{1-q^6},\cdots,$$

whence:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^4}{(1-q^2)(1-q^4)}, \quad A''' = \frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}, \cdots,$$

as it is propounded.

For the demonstration of formula (2.) I observe that the expression:

$$\frac{1}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)\cdots},$$

having put qz instead of z and multiplied by $\frac{1}{1-qz}$, remains unchanged; whence having put:

$$\begin{aligned}&\frac{1}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)\cdots}\\ &= 1 + \frac{A'z}{1-qz} + \frac{A''z^2}{(1-qz)(1-q^2z)} + \frac{A'''z^3}{(1-qz)(1-q^2z)(1-q^3z)} + \cdots,\end{aligned}$$

we obtain:

$$\begin{aligned}&1 + \frac{A'z}{1-qz} + \frac{A''z^2}{(1-qz)(1-q^2z)} + \frac{A'''z^3}{(1-qz)(1-q^2z)(1-q^3z)} + \cdots\\ &= \frac{1}{1-qz} + \frac{A'qz}{(1-qz)(1-q^2z))} + \frac{A''q^2z^2}{(1-qz)(1-q^2z)(1-q^3z)} + \frac{A'''q^3z^3}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)} + \cdots\\ &= 1 + \frac{(q+A'q)z}{1-qz} + \frac{(q^3A'+q^2A'')z^2}{(1-qz)(1-q^2z)} + \frac{(q^5A''+q^3A''')z^3}{(1-qz)(1-q^2z)(1-q^3z)} + \cdots\end{aligned}$$

Hence, it follows:

$$A' = q + A'q, \quad A'' = q^3A' + q^2A'', \quad A''' = q^5A'' + q^3A''', \cdots$$

and hence:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^3A'}{1-q^2}, \quad A''' = \frac{q^5A''}{1-q^3}, \cdots$$

whence:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^4}{(1-q)(1-q^2)}, \quad A''' = \frac{q^9}{(1-q)(1-q^2)(1-q^3)}, \cdots,$$

as it was propounded.

Now, let us form the product:

$$\left\{(1+qz)(1+q^3z)(1+q^5z)\cdots\right\}\left\{\left(1+\frac{q}{z}\right)\left(1+\frac{q^3}{z}\right)\left(1+\frac{q^5}{z}\right)\right\}$$

$$=\left\{1+\frac{q}{1-q}z+\frac{q^4}{(1-q^2)(1-q^4)}z^2+\frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}z^3+\cdots\right\}$$
$$\times\left\{1+\frac{q}{1-q}\frac{1}{z}+\frac{q^4}{(1-q^2)(1-q^4)}\frac{1}{z^2}+\frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}\frac{1}{z^3}+\cdots\right\}.$$

We find the following for the coefficient of z^n or even $\frac{1}{z^n}$ which we want to put $B^{(n)}$:

$$B^{(n)}=\frac{q^{nn}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}$$
$$\times\left\{\begin{array}{c}1+\dfrac{q^2}{1-q^2}\cdot\dfrac{q^{2n}}{1-q^{2n+2}}+\dfrac{q^8}{(1-q^2)(1-q^4)}\cdot\dfrac{q^{4n}}{(1-q^{2n+2})(1-q^{2n+4})}\\+\dfrac{q^{18}}{(1-q^2)(1-q^4)(1-q^6)}\cdot\dfrac{q^{6n}}{(1-q^{2n+2})(1-q^{2n+4})(1-q^{2n+6})}+\cdots\end{array}\right\}$$

But from formula (2.) having put q^2 instead of q and $z=q^{2n}$ which is seen in the braces we find:

$$=\frac{1}{(1-q^{2n+2})(1-q^{2n+4})(1-q^{2n+6})(1-q^{2n+8})\cdots},$$

whence:

$$B^{(n)}=\frac{q^{nn}}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

and hence:

$$\left\{(1+qz)(1+q^3z)(1+q^5q)\cdots\right\}\left\{\left(1+\frac{q}{z}\right)\left(1+\frac{q^3}{z}\right)\left(1+\frac{q^5}{z}\right)\right\}$$
$$=\frac{1+q\left(z+\frac{q}{z}\right)+q^4\left(z^2+\frac{q}{z^2}\right)+q^9\left(z+^3\frac{q}{z^3}\right)+\cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

or having put $z = e^{2ix}$ and changed q to $-q$:

$$(1 - 2q\cos 2x + q^2)(1 - 2q^3\cos 2x + q^6)(1 - 2q^5\cos 2x + q^{10})\cdots$$
$$= \frac{1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}.$$

What was to be demonstrated.

If one puts $-qz^2$ instead of z and multiplies by $\sqrt[4]{q}z$ it arises:

$$\sqrt[4]{q}\left(z - \frac{1}{z}\right)\left\{(1-q^2z^2)(1-q^4z^2)(1-q^6z^2)\cdots\right\}\left\{\left(1-\frac{q^2}{z^2}\right)\left(1-\frac{q^4}{z^2}\right)\left(1-\frac{q^6}{z^2}\right)\cdots\right\}$$
$$= \frac{\sqrt[4]{q}\left(z-\frac{1}{z}\right) - \sqrt[4]{q^9}\left(z^3 - \frac{1}{z^3}\right) + \sqrt[4]{q^{25}}\left(z^5 - \frac{1}{z^5}\right) - \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

or having put $z = e^{ix}$:

$$2\sqrt[4]{q}\sin x(1 - 2q^2\cos 2x + q^4)(1 - 2q^4\cos 2x + q^8)(1 - 2q^6\cos 2x + q^{12})\cdots$$
$$= \frac{2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

which is the other found expansion.

65.

The expansions of the functions:

$$(1.)\qquad \Theta\left(\frac{2Kx}{\pi}\right) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots$$
$$(2.)\qquad H\left(\frac{2Kx}{\pi}\right) = 2\sqrt[4]{q}\sin x - \sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - \sqrt[4]{q^{49}}\sin 7x + \cdots$$

immediately lead to a new expansion of elliptic functions. For, we obtain from formulas (1.) by putting $u = \frac{2Kx}{\pi}$:

$$\begin{aligned}
\sin\operatorname{am}\frac{2Kx}{\pi} &= \frac{1}{\sqrt{k}}\cdot\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}\\
\cos\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{\frac{k'}{k}}\cdot\frac{H\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}\\
\Delta\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{k'}\cdot\frac{\Theta\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)},
\end{aligned}$$

whence:

$$\begin{aligned}
(3.)\quad \sin\frac{2Kx}{\pi} &= \frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots}\\
(4.)\quad \cos\frac{2Kx}{\pi} &= \sqrt{\frac{k'}{k}}\cdot\frac{2\sqrt[4]{q}\cos x + 2\sqrt[4]{q^9}\cos 3x + 2\sqrt[4]{q^{25}}\cos 5x + 2\sqrt[4]{q^{49}}\cos 7x + \cdots}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots}\\
(5.)\quad \Delta\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{k'}\cdot\frac{1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+2q^{16}\cos 8x+\cdots}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots}.
\end{aligned}$$

Further from (2.), (3.) § 61 because it was put $\Theta(0)=\sqrt{\frac{2k'K}{\pi}}$ we obtain:

$$\Theta(K)=\sqrt{\frac{2K}{\pi}},\quad H(K)=\sqrt{\frac{2kK}{\pi}},\quad \Theta(0)=\sqrt{\frac{2k'K}{\pi}},\quad H'(0)=\sqrt{\frac{2kk'K}{\pi}},$$

whence from (1.), (2.):

$$\begin{aligned}
(6.)\quad & \sqrt{\frac{2K}{\pi}} && = 1+2q+2q^4+2q^9+2q^{16}+2q^{25}+\cdots\\
(7.)\quad & \sqrt{\frac{2kK}{\pi}} && = 2\sqrt[4]{q}+2\sqrt[4]{q^9}+2\sqrt[4]{q^{25}}+2\sqrt[4]{q^{49}}+2\sqrt[4]{q^{81}}+\cdots\\
(8.)\quad & \sqrt{\frac{2k'K'}{\pi}} && = 1-2q+2q^4-2q^9+2q^{16}-2q^{25}+\cdots\\
(9.)\quad & \sqrt{kk'\left(\frac{2K}{\pi}\right)^3} && = 2\sqrt[4]{q}-6\sqrt[4]{q^9}+10\sqrt[4]{q^{25}}-14\sqrt[4]{q^{49}}+18\sqrt[4]{q^{81}}-\cdots,
\end{aligned}$$

whence also:

$$(10.)\quad \sqrt{k} = \frac{2\sqrt[4]{q}+1\sqrt[4]{q^9}+2\sqrt[4]{q^{25}}+2\sqrt[4]{q^{49}}+2\sqrt[4]{q^{81}}+\cdots}{1+2q+2q^4+2q^9+2q^{16}+\cdots}$$

$$(11.)\quad \sqrt{k'} = \frac{1-2q+2q^4-2q^9+2q^{16}-2q^{25}+\cdots}{1+2q+2q^4+2q^9+2q^{16}+2q^{25}+\cdots}.$$

Further, because $Z(u) = \frac{\Theta'(u)}{\Theta(u)}$, $\Pi(u,a) = uZ(a) + \frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}$:

$$(12.)\quad \frac{2K}{\pi}\cdot Z\left(\frac{2Kx}{\pi}\right) = \frac{4q\sin 2x - 8q^4\sin 4x + 12q^9\sin 6x - 16q^{16}\sin 8x + \cdots}{1-2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots}$$

$$(13.)\quad \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right)$$

$$= \frac{2Kx}{\pi}\cdot Z\left(\frac{2KA}{\pi}\right) + \frac{1}{2}\ln\frac{1-2q\cos 2(x-A)+2q^4\cos 4(x-A)-2q^9\cos 6(x-A)+\cdots}{1-2q\cos 2(x+A)+2q^4\cos 4(x+A)-2q^9\cos 6(x+A)+\cdots}.$$

This is the third expansion of elliptic functions.

66.

From the found expansions:

$$(1.)\quad [(1-q^2)(1-q^4)(1-q^6)\cdots](1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots$$
$$= 1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots$$
$$[(1-q^2)(1-q^4)(1-q^6)\cdots]\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots$$
$$= \sin x - q^2\sin 3x + q^6\sin 5x - q^{12}\sin 7x + q^{20}\sin 9x - \cdots,$$

the second of which having put $\sqrt{q}$ instead of q can also be exhibited as this:

$$(2.)\quad [(1-q)(1-q^2)(1-q^3)\cdots]\sin(1-2q\cos 2x+q^2)(1-2q^2\cos 2x+q^4)(1-2q^3\cos 2x+q^6)\cdots$$
$$= \sin x - q\sin 3x + q^3\sin 5x - q^6\sin 7x + q^{10}\sin 9x - q^{15}\sin 11x + \cdots,$$

it follows having put $x = 0$, $x = \frac{\pi}{2}$:

$$(3.)\quad \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}{(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots} = 1-2q+2q^4-2q^9+2q^{16}-\cdots$$

$$(4.)\quad \frac{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots} = 1+q+q^3+q^6+q^{10}+q^{15}+\cdots$$

$$(5.)\quad [(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots]^3 = 1-3q+5q^3-7q^6+9q^{10}-\cdots$$

Let us in (2.) put $x = \frac{\pi}{3}$, it is $\sin x = +\sqrt{\frac{3}{4}}$, $\sin 3x = 0$, $\sin 5x = -\sqrt{\frac{3}{4}}$, $\sin 7x = +\sqrt{\frac{3}{4}}$, etc.; further, $(1-q)(1-2q\cos 3x + q^2) = 1 - q^3$, whence (2.) goes over into this formula:

$$(1-q^3)(1-q^6)(1-q^9)(1-q^{12})\cdots = 1 - q^3 - q^6 + q^{15} + q^{21} - q^{36} - \cdots$$

or:

$$(6.)\qquad (1-q)(1-q^2)(1-q^3)(1-q^4)\cdots = 1 - q - q^2 + q^5 + q^7 - q^{12} - \cdots,$$

the general term of which series is:

$$(-1)^n q^{\frac{3nn\pm n}{2}}.$$

Having compared (5.), (6.) we obtain:

$$(7.)\quad [1 - q - q^2 + q^5 + q^7 - q^{12} - \cdots]^3 = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \cdots.$$

Formula (4.) was also found by Gauss in the paper: *Summatio serierum quarundam singularium.* Comm. Gott. Vol. I. a. 1808-1811. He deduced it from the following memorable formula:

$$(8.)\qquad \frac{(1-qz)(1-q^3z)(1-q^5z)(1-q^7z)\cdots}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots}$$
$$= 1 + \frac{q(1-z)}{1-q} + \frac{q^3(1-z)(1-qz)}{(1-q)(1-q^2)} + \frac{q^6(1-z)(1-qz)(1-q^2z)}{(1-q)(1-q^2)(1-q^3)} + \cdots,$$

having put $z = q$. To these one can add other similar formulas whose proof I omit here:

$$(9.)\qquad \frac{1}{2}\frac{(1+z)(1+qz)(1+q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots} + \frac{1}{2}\frac{(1-z)(1-qz)(1-q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$
$$= 1 - \frac{q(1-z^2)}{1-q^2} + \frac{q^4(1-z^2)(1-q^2z^2)}{(1-q^2)(1-q^4)} - \frac{q^9(1-z^2)(1-q^2z^2)(1-q^4z^2)\cdots}{(1-q^2)(1-q^4)(1-q^6)} + \cdots$$

$$(10.)\qquad \frac{q}{2z}\frac{(1+z)(1+qz)(1+q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots} - \frac{q}{2z}\frac{(1-z)(1-qz)(1-q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$

$$= q - \frac{q^4(1-z^2)}{1-q^2} + \frac{q^9(1-z^2)(1-q^2z^2)}{(1-q^2)(1-q^4)} - \frac{q^{16}(1-z^2)(1-q^2z^2)(1-q^4z^2)\cdots}{(1-q^2)(1-q^4)(1-q^6)} + \cdots$$

of which (9.) having put $z = q$ yields:

$$\frac{1}{2} + \frac{1}{2}\frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots} = 1 - q + q^4 - q^9 + \cdots$$

or:

$$\frac{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}{(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots} = 1 - 2q + 2q^4 - 2q^9 + \cdots,$$

which is formula (3.).

Formula (6.) which is of highest depth as the one depending on the trisection of elliptic functions was already found by Euler a long time ago and lucently proved. This extraordinary proof is to be treated on another occasion in more detail.

To these we add the following expansions:

$$(11.) \qquad \frac{\sqrt{kk'\left(\frac{2K}{\pi}\right)^3}}{\Theta\left(\frac{2kx}{\pi}\right)} = \frac{2\sqrt[4]{q}[(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots]^2}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}$$

$$= \frac{2\sqrt[4]{q}(1-q^2)}{1-2q\cos 2x+q^2} - \frac{2\sqrt[4]{q^9}(1-q^6)}{1-2q^3\cos 2x+q^6} + \frac{2\sqrt[4]{q^{25}}(1-q^{10})}{1-2q^5\cos 2x+q^{10}} - \cdots$$

$$(12.) \qquad \frac{\sqrt{kk'\left(\frac{2K}{\pi}\right)^3}}{H\left(\frac{2kx}{\pi}\right)} = \frac{[(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots]^2}{\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}$$

$$= \frac{1}{\sin x} - \frac{4q^2(1+q^2)\sin x}{1-2q^2\cos 2x+q^4} + \frac{4q^6(1+q^4)\sin x}{1-2q^4\cos 2x+q^8} - \frac{4q^{12}(1+q^6)\sin x}{1-2q^6\cos 2x+q^{12}} + \cdots$$

$$= \frac{1}{\sin x}\left\{\frac{(1-q^2)(1-q^4)}{1-2q^2\cos 2x+q^4} - \frac{q^2(1-q^4)(1-q^8)}{1-2q^4\cos 2x+q^8} + \frac{q^6(1-q^6)(1-q^{12})}{1-2q^6\cos 2x+q^{12}} - \cdots\right\},$$

which is easily obtained from the known theory of composite fractions into simple ones.

Hence, one deduces the special expansions:

$$(13.)\qquad \frac{2kK}{\pi} = 4\sqrt{q}\left(\frac{1+q^2}{1-q^2}\right) - 4\sqrt{q^9}\left(\frac{1+q^6}{1-q^6}\right) + 4\sqrt{q^{25}}\left(\frac{1+q^{10}}{1-q^{10}}\right) - \cdots$$

$$(14.)\qquad \frac{2k'K}{\pi} = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^2} - \frac{4q^6}{1+q^3} + \frac{q4^{10}}{1+q^4} - \cdots$$

Having compared these expansions of the expressions $\frac{2kK}{\pi}$, $\frac{2k'K}{\pi}$ exhibited above it arises:

$$\frac{\sqrt{q}}{1-q} - \frac{\sqrt{q^3}}{1-q^3} + \frac{\sqrt{q^5}}{1-q^5} - \frac{\sqrt{q^7}}{1-q^7} + \cdots = \sqrt{q}\left(\frac{1+q^2}{1-q^2}\right) - \sqrt{q^9}\left(\frac{1+q^6}{1-q^6}\right) + \sqrt{q}\left(\frac{1+q^{10}}{1-q^{10}}\right) - \cdots$$

$$1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^3} - \frac{4q^5}{1+q^5} + \frac{4q^7}{1+q^7} - \cdots = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^2} - \frac{4q^6}{1+q^3} + \frac{4q^{10}}{1+q^4} - \cdots$$

In similar manner, Clausen recently observed that the series:

$$\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \frac{q^4}{1-q^4} + \cdots$$

can be transformed into:

$$q\left(\frac{1+q}{1-q}\right) + q^4\left(\frac{1+q^2}{1-q^2}\right) + q^9\left(\frac{1+q^3}{1-q^3}\right) + q^{16}\left(\frac{1+q^4}{1-q^4}\right) + \cdots$$

Above we found the expansions of $\frac{2K}{\pi}$, $\frac{2kK}{\pi}$ and their second, third, fourth powers into series. Therefore, these yield expansions of the second, fourths, sixth and eighth powers of the expressions:

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots$$

$$\sqrt{\frac{2kK}{\pi}} = 2\sqrt[4]{q} + 2\sqrt[4]{q^9} + 2\sqrt[4]{q^{25}} + 2\sqrt[4]{q^{49}} + \cdots$$

whence various arithmetical theorems follow. So, for the sake of an example, from the formula:

$$\begin{aligned}\left(\frac{2K}{\pi}\right)^2 &= \left\{1+2q+2q^4+2q^9+2q^{16}+\cdots\right\}^4\\ &= 1+8\left\{\frac{q}{1-q}+\frac{2q^2}{1+q^2}+\frac{3q^3}{1-q^3}+\frac{4q^4}{1+q^4}+\cdots\right\}\\ &= 1+8\sum\varphi(p)\left\{q^p+3q^{2p}+3q^{4p}+3q^{8p}+\cdots\right\},\end{aligned}$$

where $\varphi(p)$ is an arbitrary odd number, $\varphi(p)$ is the sum of factors of p, the infamous Fermat-Theorem follows as a corollary, namely, that any number is the sum of four squares.